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# On the phase transition in certain percolation models

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematics

October 2015

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Christopher James Edwin Daniels

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## Summary

Consider random sequential adsorption on a chequerboard lattice with arrivals at rate 1 on light squares and at rate  $\lambda$  on dark squares. Ultimately, each square is either occupied, or blocked by an occupied neighbour. Colour the occupied dark squares and blocked light sites *black*, and the remaining squares *white*. Independently at each meeting-point of four squares, allow diagonal connections between black squares with probability  $p$ ; otherwise allow diagonal connections between white squares.

We show that there is a critical surface of pairs  $(\lambda, p)$ , containing the pair  $(1, 0.5)$ , such that for  $(\lambda, p)$  lying above (respectively, below) the critical surface the black (resp. white) phase percolates, and on the critical surface neither phase percolates.

We find conditions satisfied by a broad class of essentially planar percolation models such that for a model satisfying the conditions, the presence or absence of percolation is determined by what happens in a collection of finite boxes.

This criterion applies to a (non-degenerate) Poisson Boolean model, to the random connection model for some sufficiently high  $p < 1$ , and to the model described above.

We also find conditions that do not require rotation invariance which produce a comparable result; these conditions seem plausible, but finding non-trivial examples is a matter for further research.

# Chapter 1

## Introduction

A percolation model on a space  $X$  is a method of generating a collection of random subsets of  $X$ . For many percolation models we produce only one subset (of potentially infinitely many components), while in other percolation models we may produce multiple or infinitely many subsets. We then say that a model percolates if one of these subsets contains an unbounded region.

A percolation model will generally depend upon one or more parameters, and we can thus consider the probability that a model percolates for specific values of the parameters; if the model is increasing in its parameters, we will frequently be able to find some 'critical' parameters such that for a realization of the model with parameters strictly greater than the 'critical' parameters there is a positive probability that the model percolates, and for a realization of the model with parameters strictly less than the 'critical' parameters the model does not percolate almost surely.

First, we consider random sequential adsorption (abbreviated RSA throughout this paper). Random sequential adsorption is a term for a family of probability models for irreversible particle deposition. Particles arrive at random locations and times onto a surface, and if accepted a particle blocks nearby locations on the surface from accepting future arrivals. We consider a discrete version of RSA on the initially empty integer lattice  $\mathbb{Z}^2$ , with the arrival time at a lattice site  $x$  given by an exponential random variable  $T_x$  with parameter  $\lambda_x$ , with  $(T_x)_{x \in \mathbb{Z}^2}$  independent. An arrival at the site  $x$  blocks future arrivals at all sites adjacent to  $x$  (that is, sites  $y$  such that  $|x - y| = 1$  where  $|\cdot|$  denotes the Euclidean norm). If  $\sup_x \lambda_x < \infty$  this model is well defined; see [20]. On this lattice we define the

even (respectively, odd) sites to be those at an even (respectively, odd) graph distance from the origin.

Ultimately, all sites will either have accepted a particle or be blocked. The distribution of the occupied and blocked sites in this ultimate state is called the jamming distribution; under the jamming distribution the sites of  $\mathbb{Z}^2$  could then be divided into an even phase and odd phase, where the even phase consists of occupied even sites and blocked odd sites. Site percolation of the even phase was considered in [21], in the case where for some  $\lambda > 0$  we have  $\lambda_x = 1$  for odd  $x$  and  $\lambda_x = \lambda$  for even  $x$ .

Penrose and Rosoman [21] proved that the critical parameter  $\lambda$  for RSA on the integer lattice  $\mathbb{Z}^2$  is strictly greater than 1. The proof of this uses an enhanced RSA (denoted eRSA below) model on a new lattice called  $\Lambda$  throughout this paper. We associate with each site  $x \in \mathbb{Z}^2$  a site  $x' := x + (1/2, 1/2)$ . The lattice  $\Lambda$  has vertex set  $\cup_{x \in \mathbb{Z}^2} \{x, x'\}$ , with an edge between sites  $x \in \mathbb{Z}^2$  and  $y \in \mathbb{Z}^2$  if  $|x - y| = 1$ , and an edge between  $x'$  and  $y$  if  $|x' - y| = \frac{\sqrt{2}}{2}$  (here  $|\cdot|$  is the Euclidean distance). We refer to the added sites  $x'$  as diamond sites, and the original sites  $x$  as octagon sites; each octagon site has degree 8 and each diamond site has degree 4 (see Fig 2-1).

We introduce a parameter  $p \in [0, 1]$ . Each of the diamond sites  $x'$  is independently taken to be in the even phase with probability  $p$ , and otherwise in the odd phase. Considering percolation on this new lattice, where a site is considered black if it is in the even phase and otherwise white, we say the even (resp. odd) phase percolates if there is an infinite component of black (resp. white) sites.

We investigate the critical behaviour of eRSA using a method introduced by Bollobás and Riordan [2] for use in proving that for random Voronoi tilings of the plane,  $p_c = 1/2$ . The same method has been used to investigate and find a critical parameter for several models (such as [3], [5], [12], [19])

This method has essentially 3 steps. First, argue from symmetry conditions or otherwise that you have some parameters such that the probability of crossing a box and of failing to cross the same box are equal, regardless of the size of the box (all that is required is that the probability be bounded away from 0 and well away from 1). Second, through use of a sharp threshold result demonstrate that increasing the parameters slightly will result in the long way crossing of a sequence of rectangles of fixed length ratio to occur with high probability. Finally,

couple the model to an edge percolation model with finite range dependence.

While eRSA easily satisfies the required symmetry conditions, we need to expand upon an existing sharp thresholds result in order to apply this method; doing so forms Section 2.2. Having achieved this result, we then apply the method to eRSA and arrange inequalities to demonstrate that for fixed  $0 < p < 1$  there is some  $\lambda_c(p) > 0$  with a sharp phase transition; that is that for  $\lambda < \lambda_c$  there is percolation of the odd phase, for  $\lambda > \lambda_c$  there is percolation of the even phase, and that for  $\lambda = \lambda_c$  neither phase percolates.

We then present an alternate method for demonstrating that eRSA does not percolate at criticality inspired by Duminil-Copin et al. [9], and a broad set of conditions such that any percolation model which satisfies the conditions and has the probability of finite box events continuous in the parameters of the model likewise does not percolate at criticality.

The planar Poisson Boolean model is a simple continuum percolation model in which we distribute points on the plane at rate  $\lambda$ , and then independently deposit a shape at each of the resulting points, either considering percolation in the the area covered, or percolation in the set of points where two points are connected if the associated shapes have nonempty intersection. The model has been studied in great detail and in generality (see for instance [17], [15], [11])

We consider the Poisson Boolean model where the shapes deposited are balls of random radius with distribution  $X$ , and demonstrate that for all  $X$  with  $\mathbb{E}[X^2] < \infty$  the conditions are satisfied and there is no percolation at  $\lambda_c$ .

We also investigate a simple model which lacks the geometric properties relied upon in the method of Bollobás and Riordan. The random connection model with probability  $p < 1$  is generated by placing points on the plane according to a Poisson point process of rate  $\lambda$ , and then for any two points at distance no more than 1 from each other connecting them with probability  $p$ . We show that there is some  $p' < 1$  such that for all  $p > p'$  there is some  $\lambda_c$  such that for  $\lambda \leq \lambda_c$  the random connection model with rate  $\lambda$  and connection probability  $p$  does not percolate, and that for  $\lambda > \lambda_c$  the random connection model with rate  $\lambda$  and connection probability  $p$  does percolate. Further details on this model can be found in [17] and [10]

Finally, we produce a set of conditions that do not assume rotation invariance such that once more any model which satisfies all the conditions and has the



probability of events depending only on a finite box continuous in the parameters of the model likewise has that there is no percolation at the critical parameters.

## Chapter 2

# Enhanced Random Sequential Adsorption

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This chapter is also available as [8], with minor alterations.

## 2.1 Outline

The enhancement used in eRSA adds a level of symmetry to the model RSA, ensuring that any given rectangle possesses either a horizontal black crossing or vertical white crossing. Taking the special case with  $p = 1$  amounts to always allowing diagonal connections between black sites of  $\mathbb{Z}^2$ , and never allowing diagonal connections between white sites. Taking  $p = 0$  amounts to the opposite.

In this chapter we consider the enhanced model in its own right, with a further parameter  $\lambda \in \mathbb{R}_+ := (0, \infty)$  and with  $\lambda_x = 1$  for odd  $x$  and  $\lambda_x = \lambda$  for even  $x$ . For  $p \in [0, 1]$  we define the critical values

$$\begin{aligned}\lambda_c^+(p) &:= \inf\{\lambda : \text{the even phase percolates in eRSA}(\lambda, p)\}; \\ \lambda_c^-(p) &:= \sup\{\lambda : \text{the odd phase percolates in eRSA}(\lambda, p)\}.\end{aligned}$$

It is natural to ask whether these values coincide, and if so, to try to understand the behaviour of the critical surface in  $(\lambda, p)$  space for this model; for example, the symmetry suggests the pair  $(\lambda = 1, p = 1/2)$  should be critical. Our main result provides some information on these issues.

**Theorem 2.1.1.** *(i) For any  $p \in [0, 1]$  we have  $\lambda_c^-(p) \leq \lambda_c^+(p)$ , with equality whenever  $0 < p < 1$ .*

*(ii) For any  $p \in [0, 1]$  there is no percolation of the even phase for eRSA with parameters  $(\lambda_c^+(p), p)$ , and no percolation of the odd phase for eRSA with parameters  $(\lambda_c^-(p), p)$ .*

*(iii) It is the case that  $\lambda_c^+(1/2) = 1$ .*

*(iv) For any  $\varepsilon \in (0, 1/2)$ , the functions  $\lambda_c^+ : [\varepsilon, 1] \rightarrow \mathbb{R}_+$  and  $\lambda_c^- : [0, 1 - \varepsilon] \rightarrow \mathbb{R}_+$  are strictly decreasing and Lipschitz, and the inverse of the function  $\lambda_c^+ : [\varepsilon, 1 - \varepsilon] \rightarrow [\lambda_c^+(1 - \varepsilon), \lambda_c^+(\varepsilon)]$  is also strictly decreasing and Lipschitz.*

We conjecture that  $\lambda_c^+(p) = \lambda_c^-(p)$  for all  $p \in [0, 1]$  but we prove this only for  $p \in (0, 1)$ . It is clear from the theorem that the inverse function of  $\lambda_c^+(\cdot)$  is the function  $p_c^+(\cdot)$  defined by

$$p_c^+(\lambda) := \inf\{p : \text{the even phase percolates in eRSA}(\lambda, p)\}.$$

For the proof, most of the work goes into showing that if the *odd* phase

does *not* percolate at a certain  $(\lambda, p)$  (Assumption A), then after an arbitrarily small increase in either  $\lambda$  or  $p$  the *even* phase *does* percolate (Conclusion B). The strategy to prove this is similar to a method used by Bollobás and Riordan in [2] to prove that the critical value for Voronoi percolation in the plane is  $1/2$ , and goes as follows. Under Assumption A, we shall adapt known methods to deduce that the even phase crosses an arbitrarily large rectangle of aspect ratio 3 the long way, with non-vanishing probability. Then using a suitable sharp thresholds result for increasing events in a finite product space (presented in Section 2.2, and perhaps of independent interest), we shall deduce that after increasing  $\lambda$  or  $p$  we have a crossing of such a rectangle with probability close to 1, and then a standard comparison with 1-dependent percolation yields Conclusion B.

The main elements in the proof are as follows. First, we derive a sharp thresholds result that is suitable for our purposes. Second, we discretize time so as to be able to use this result. Third, we demonstrate that the approximation error involved in the discretization can be compensated for with a slight increase in parameters that vanishes as the size of the rectangle approaches infinity.

## 2.2 A sharp thresholds result

The *sharp threshold property* [14] for increasing events in  $\{0, 1\}^n$  says that for any such event and any fixed  $\eta \in (0, 1/2)$ , when  $n$  is large the threshold value of  $p$  above which the probability of such an event (under product measure with parameter  $p$ ) exceeds  $1 - \eta$ , is only slightly larger than the corresponding threshold for the event to have probability at least  $\eta$ .

In Proposition 2.2.1 below, we present a similar threshold result for events in  $\{0, 1, \dots, k\}^n$  for any fixed  $k$ , satisfying a symmetry assumption. Such a result was given in [6] for the case  $k = 2$ ; we adapt this to general  $k$  and give a more detailed proof than that of [6].

Let  $k \in \mathbb{N}$ . For  $n, m \in \mathbb{N}$ , a subset  $E \subset \{0, 1, \dots, k\}^n$  is said to have symmetry of order  $m$  if there is a group action on  $[n] := \{1, 2, \dots, n\}$  in which each orbit has size at least  $m$ , such that the induced action on  $\{0, 1, \dots, k\}^n$  preserves  $E$ ; for instance, if  $n$  is even then a subset  $E \subset \{0, 1, \dots, k\}^{n^2}$  which is preserved by even translations of the  $n$  by  $n$  torus  $[n] \times [n]$  (identified with  $[n^2]$ ) would have symmetry of order  $n^2/2$ .

Additionally, let  $\underline{p} = (p_0, p_1, \dots, p_k)$  be a probability vector (i.e., a vector with nonnegative entries summing to 1). Define

$$\beta_{\underline{p}}(x) := \max \left\{ j \in \{0, \dots, k\} : \sum_{i=0}^{j-1} p_i \leq x \right\}, \quad x \in [0, 1).$$

where by definition we set  $\sum_{i=0}^{-1} p_i = 0$ . Define  $p_{\max}(\underline{p})$  to be the *second largest* of the numbers  $p_0, p_1, \dots, p_k$

We write  $\mathbb{P}_{\underline{p}}$  for the probability measure on  $\{0, \dots, k\}$  with probability mass function  $\underline{p}$ , and for  $n \in \mathbb{N}$  we write  $\mathbb{P}_{\underline{p}}^n$  for the  $n$ -fold product of this probability measure (a probability measure on  $\{0, \dots, k\}^n$ ). We say that  $E \subset \{0, 1, \dots, k\}^n$  is *increasing*, if for every  $x = (x_1, \dots, x_k)$  and  $y = (y_1, \dots, y_k)$  in  $\{0, 1, \dots, k\}^n$  such that  $x \in E$  and  $y_i \geq x_i$  for  $i \in \{0, 1, \dots, n\}$ , we have  $y \in E$ .

Given probability vectors  $\underline{p} = (p_0, \dots, p_k)$  and  $\underline{q} = (q_0, \dots, q_k)$ , we say that  $\underline{q}$  *dominates*  $\underline{p}$  if for  $j = 0, 1, 2, \dots, k-1$  we have  $\sum_{i=0}^j (p_i - q_i) \geq 0$ . Note that  $\underline{q}$  dominates  $\underline{p}$ , if and only if there are coupled random variables  $X, Y$  taking values in  $\{0, 1, \dots, k\}$  such that  $X$  has distribution  $\mathbb{P}_{\underline{p}}$  and  $Y$  has distribution  $\mathbb{P}_{\underline{q}}$  and  $Y \geq X$  almost surely. Thus, if  $\underline{q}$  dominates  $\underline{p}$  then  $\mathbb{P}_{\underline{q}}^n(E) \geq \mathbb{P}_{\underline{p}}^n(E)$  for any  $n \in \mathbb{N}$  and any increasing  $E \subset \{0, 1, \dots, k\}^n$ .

**Proposition 2.2.1.** *Let  $k, n, m, \in \mathbb{N}$ , let  $\eta \in (0, 1/2)$  and let  $\gamma > 0$ . Suppose  $\underline{p} = (p_0, p_1, \dots, p_k)$  and  $\underline{q} = (q_0, q_1, \dots, q_k)$  are probability vectors such that  $p_0 \geq \gamma$ ,  $p_k \leq 1 - \gamma$  and  $\underline{q}$  dominates  $\underline{p} + (-\gamma, 0, \dots, 0, \gamma)$ . Let  $q_{\max}$  denote the second largest of the numbers  $p_0, \dots, p_{k-1}, p_k + \gamma$ , and suppose also that*

$$\gamma \log m \geq 200k^2 \log(1/\eta) q_{\max} \log(4/q_{\max}). \quad (2.2.1)$$

*Then for any increasing  $E \subset \{0, 1, \dots, k\}^n$  with symmetry of order  $m$ , and with  $\mathbb{P}_{\underline{p}}^n(E) > \eta$ , we have  $\mathbb{P}_{\underline{q}}^n(E) > 1 - \eta$ .*

The remainder of this section is devoted to proving this, via a series of lemmas.

Given  $\ell \in \mathbb{N}$ , let  $h_\ell : [0, 1) \rightarrow [0, 1)$  be the function which inverts the  $\ell$ th digit of the binary expansion of a number (using the terminating expansion wherever there is a choice). Now we let  $U$  be a uniform  $(0, 1)$  distributed random variable,

and for  $f : \{0, 1, \dots, k\} \rightarrow \{0, 1\}$  define

$$w_{\ell, \underline{\mathbf{p}}}(f) = \mathbb{P} \left[ f \circ \beta_{\underline{\mathbf{p}}}(U) \neq f \circ \beta_{\underline{\mathbf{p}}}(h_{\ell}(U)) \right];$$

$$w_{\underline{\mathbf{p}}}(f) = \sum_{\ell=1}^{\infty} w_{\ell, \underline{\mathbf{p}}}(f).$$

The following refinement of Lemma 3 of [7] was noted with a sketch proof in the proof of Theorem 3.1 of [14]. We give a more detailed proof here.

**Lemma 2.2.1.** *For any function  $f : \{0, 1\} \rightarrow \{0, 1\}$  and probability vector  $\underline{\mathbf{p}} = (p_0, p_1)$ , letting  $p := \min(p_0, p_1)$  we have that*

$$w_{\underline{\mathbf{p}}}(f) \leq 2p \log_2(4/p). \quad (2.2.2)$$

Moreover  $2p \log_2(4/p)$  is increasing in  $p$  for  $p \in (0, 1)$ .

*Proof.* Assume that  $f(0) \neq f(1)$ . Then with  $U, \ell$  as above,

$$\begin{aligned} w_{\ell, \underline{\mathbf{p}}}(f) &= 2\mathbb{P}(U < p_0 < h_{\ell}(U)) \\ &\leq 2 \min(\mathbb{P}(U < p_0), \mathbb{P}(h_{\ell}(U) > p_0), \mathbb{P}(p_0 - 2^{-\ell} \leq U < p_0)) \\ &\leq \min(2^{1-\ell}, 2p). \end{aligned}$$

With this inequality we obtain that

$$\begin{aligned} w_{\underline{\mathbf{p}}}(f) &= \sum_{\ell=1}^{\infty} w_{\ell, \underline{\mathbf{p}}}(f) \leq 2 \sum_{\ell=1}^{\infty} \min(2^{-\ell}, p) \\ &\leq 2 \left( \sum_{\ell=1}^{\lfloor \log_2(1/p) \rfloor} p + \sum_{\ell=\lfloor \log_2(1/p) \rfloor + 1}^{\infty} 2^{-\ell} \right) \\ &\leq 2 \left( p \lfloor \log_2(1/p) \rfloor + 2^{-\lfloor \log_2(1/p) \rfloor} \right) \\ &\leq 2p \log_2(1/p) + 4p = 2p \log_2(4/p) \end{aligned}$$

which gives us (2.2.2). The monotonicity in  $p$  of  $p \log(4/p)$  follows by routine calculus.  $\square$

**Lemma 2.2.2.** *Let  $k \in \mathbb{N}$ . Then for any probability vector  $\underline{\mathbf{p}} = (p_0, p_1, \dots, p_k)$*

with  $p_i > 0$  for all  $i$ , and any function  $f : \{0, 1, \dots, k\} \rightarrow \{0, 1\}$ , we have

$$w_{\underline{p}}(f) \leq 3k^2 p_{\max}(\underline{p}) \log(4/p_{\max}(\underline{p})).$$

*Proof.* For  $0 \leq j \leq k-1$  we define  $\underline{q}_j := (\sum_{\ell=0}^j p_\ell, \sum_{\ell=j+1}^k p_\ell)$  and  $q_j = \sum_{\ell=0}^j p_\ell$ . Now we claim that

$$w_{i, \underline{p}}(f) \leq \sum_{j=0}^{k-1} \mathbb{P}(U < q_j < h_i(U)).$$

In order to see this we observe that for  $x < y$  we have  $\beta_{\underline{q}_j}(x) \neq \beta_{\underline{q}_j}(y)$  precisely when  $x \leq \sum_{\ell=0}^j p_\ell < y$  for some  $j$ . Letting  $g : \{0, 1\} \rightarrow \{0, 1\}$  be the identity function, we then observe that

$$w_{i, \underline{p}}(f) \leq \sum_{j=0}^{k-1} w_{\underline{q}_j}(g). \quad (2.2.3)$$

By Lemma 2.2.1 we have for  $j = 0, 1, \dots, k$  that

$$w_{\underline{q}_j}(g) \leq 2 \sum_{\ell=j+1}^k p_\ell \log_2 \left( \frac{4}{\sum_{\ell=j+1}^k p_\ell} \right) \quad (2.2.4)$$

and

$$w_{\underline{q}_j}(g) \leq 2 \sum_{\ell=0}^j p_\ell \log_2 \left( \frac{4}{\sum_{\ell=0}^j p_\ell} \right). \quad (2.2.5)$$

We have by (2.2.3) that

$$\begin{aligned} w_{\underline{p}}(f) &\leq \sum_{i=1}^{\infty} \sum_{j=0}^{k-1} w_{i, \underline{q}_j}(g) \\ &= \sum_{j=0}^{k-1} w_{\underline{q}_j}(g). \end{aligned}$$

Choose  $s \in \{0, 1, \dots, k\}$  such that  $p_s = \max(p_0, p_1, \dots, p_k)$ . Using (2.2.4) for

$j \geq s$  and (2.2.5) for  $j < s$ , we obtain

$$\begin{aligned} w_{\underline{\mathbf{p}}}(f) &\leq \left( 2 \sum_{j=s}^{k-1} \sum_{\ell=j+1}^k p_{\ell} \log_2(4/p_{\ell}) \right) + 2 \sum_{j=0}^{s-1} \sum_{\ell=0}^j p_{\ell} \log_2(4/p_{\ell}) \\ &\leq k(k+1)p_{\max}(\underline{\mathbf{p}}) \log_2(4/p_{\max}(\underline{\mathbf{p}})). \end{aligned}$$

Since  $k(k+1) \leq 3k^2$ , the result (2.2.2) follows.  $\square$

Given  $n \in \mathbb{N}$ , given  $f : \{0, 1, \dots, k\}^n \rightarrow \{0, 1\}$  and  $j \in \{0, \dots, k\}$ , and given  $x = (x_1, \dots, x_n) \in \{0, 1, \dots, k\}^n$ , we say the  $j$ th coordinate of  $x$  is *pivotal* for  $f$  if there exists  $y = (y_1, \dots, y_n) \in \{0, 1, \dots, k\}^n$ , with  $y_i = x_i$  for all  $i \neq j$ , such that  $f(x) \neq f(y)$ . Given also a probability vector  $\underline{\mathbf{p}} = (p_0, \dots, p_k)$  we define the *influence*  $I_{f, \underline{\mathbf{p}}}(j)$  of the  $j$ th coordinate on  $f$  as the probability that the  $j$ th coordinate of  $X$  is pivotal for  $f$ , where here  $X$  is a random element of  $\{0, 1, \dots, k\}^n$  with distribution  $\mathbb{P}_{\underline{\mathbf{p}}}^n$ .

**Lemma 2.2.3.** *Let  $k \in \mathbb{N}$ . For any probability vector  $\underline{\mathbf{p}} = (p_0, p_1, \dots, p_k)$  with all  $p_i > 0$ , any  $n \in \mathbb{N}$ , any function  $f : \{0, 1, \dots, k\}^n \rightarrow \{0, 1\}$ , any  $q \in [p_{\max}(\underline{\mathbf{p}}), 1]$  and any  $a \in (0, 1/16]$ , if*

$$I_{f, \underline{\mathbf{p}}}(j) \leq a q^2 (\log(4/q))^2, \quad \forall j \in [n], \quad (2.2.6)$$

*then setting  $t = \mathbb{P}_{\underline{\mathbf{p}}}^n(f^{-1}(1)) = \mathbb{E}f(X)$ , we have that*

$$\sum_{j=1}^n I_{f, \underline{\mathbf{p}}}(j) \geq \frac{t(1-t) \log(1/a)}{24k^2 q \log(4/q)}. \quad (2.2.7)$$

*Proof.* The proof is similar to that of Lemma 2 of [6]. However, the argument given there is quite sketchy, and ‘not intended to be read on its own’; it relies on arguments from Theorems 3.1 and 3.4 of [14], and both of these papers rely heavily on arguments from [7], which is itself rather concise. Moreover, none of these papers is entirely free of minor errors, which does not aid readability. Therefore to make this presentation more self-contained, and also to give explicit constants in the bounds, we think it worthwhile to give a detailed proof.

By a continuity argument, it suffices to prove the result for the case where all entries of  $\underline{\mathbf{p}}$  are dyadic rationals, i.e. to show that for any  $n, f, a$  and any  $\underline{\mathbf{p}}$  with



all entries dyadic rationals satisfying (2.2.6) for all  $j \in [n]$  we have (2.2.7).

Choose such a  $\underline{p}$  and choose  $m \in \mathbb{N}$  such that all entries of  $2^m \underline{p}$  are integers. Let  $\mathbb{Y}$  be the space  $\{0, 1\}^m$  with the uniform distribution. We identify the space  $\mathbb{X} := \{0, 1, \dots, k\}$  under measure  $\mathbb{P}_{\underline{p}}$ , with the space  $\mathbb{Y}$ , as follows. Define a function  $\tau : \mathbb{Y} \rightarrow \mathbb{X}$  as follows: the first  $2^m p_0$  elements of  $\mathbb{Y}$  (under the upwards lexicographic ordering) are mapped to  $0 \in \mathbb{X}$ , the next  $2^m p_1$  elements of  $\mathbb{Y}$  are mapped to  $1 \in \mathbb{X}$ , and so on.

Using this identification, any function  $g : \mathbb{X} \rightarrow \{0, 1\}$  induces another function  $\tilde{g} : \mathbb{Y} \rightarrow \{0, 1\}$ , given by  $\tilde{g} = g \circ \tau$ . Moreover, for  $j \in [m]$  the influence of the  $j$ th coordinate of a uniform random element of  $\mathbb{Y}$  on  $\tilde{g}$  is equal to  $w_{j, \underline{p}}(g)$ , since switching the  $j$ th digit of the binary expansion of  $U$  amounts to switching the  $j$ th component of the corresponding random element of  $\mathbb{Y}$ . Writing  $w(\tilde{g})$  for the sum (over  $j$ ) of these influences, we have by Lemma 2.2.2 that

$$w(\tilde{g}) \leq 3k^2 p_{\max}(\underline{p}) \log(4/p_{\max}(\underline{p})) \leq 3k^2 q \log(4/q). \quad (2.2.8)$$

We identify  $\mathbb{Y}$  with the power set of  $[m]$  in the natural way. For  $S \in \mathbb{Y}$  (i.e. for  $S \subset [m]$ ), we set

$$u_S(A) = (-1)^{|S \cap A|}, \quad A \subset [m].$$

It is well known (and not hard to prove) that the functions  $u_S, S \subset [m]$  form an orthonormal basis of the  $2^m$ -dimensional vector space of functions from  $\mathbb{Y}$  to  $\mathbb{R}$ , endowed with the inner product  $\langle \cdot, \cdot \rangle$  given by

$$\langle g, h \rangle = 2^{-m} \sum_{A \subset [m]} g(A)h(A).$$

Given functions  $h$  and  $g$  from  $\mathbb{Y}$  to  $\mathbb{R}$ , define the convolution  $h * g$  by

$$h * g(S) = 2^{-m} \sum_{A \subset [m]} h(A)g(S \triangle A), \quad S \subset [m], \quad (2.2.9)$$

where  $\triangle$  denotes symmetric difference. Also define the Walsh-Fourier transform  $\hat{h}$  of  $h$  by

$$\hat{h}(S) = \langle h, u_S \rangle, \quad S \subset [m]. \quad (2.2.10)$$

Associated with this is the Walsh-Fourier expansion of  $h$ , namely  $h = \sum_S \hat{h}(S)u_S$ , and the Parseval equation  $\|h\|_2^2 := \langle h, h \rangle = \sum_S \hat{h}(S)^2$ . These are both immediate from the fact that the  $u_S$  form an orthonormal basis. It is well known (and not hard to prove) that for  $S \subset [m]$  we have

$$\widehat{h * g}(S) = \hat{h}(S)\hat{g}(S). \quad (2.2.11)$$

Define  $T : \mathbb{Y} \rightarrow \mathbb{R}$  by  $T(Z) = \sum_S u_S(Z)|S|^{1/2}$ , for  $Z \subset [m]$ , where the sum is over all  $S \subset [m]$ . Then  $\hat{T}(S) = |S|^{1/2}$  for all  $S$ . Hence by (2.2.11), for any  $h : \mathbb{Y} \rightarrow \{0, 1\}$  we have  $\widehat{T * h}(S) = \hat{h}(S)|S|^{1/2}$ . Hence by the Parseval identity,

$$\|T * h\|_2^2 = \sum_{S \subset [m]} \hat{h}(S)^2 |S| = (1/4)w(h), \quad (2.2.12)$$

where  $w(h)$  is as in (2.2.8) and for the last equality we have used the first paragraph of [13, p.73].

Let  $i \in [n]$ . For  $S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n \in \mathbb{Y}$  we define the function  $h = h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n] : \mathbb{Y} \rightarrow \{0, 1\}$  by

$$h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n](S) = \tilde{f}(S_1, \dots, S_{i-1}, S, S_{i+1}, \dots, S_n), \quad (2.2.13)$$

where we set  $\tilde{f}(S_1, \dots, S_n) := f(\tau(S_1), \dots, \tau(S_n))$ . Also, define the function  $v = v[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n] : \mathbb{Y} \rightarrow \mathbb{R}$  by

$$v[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n] = T * h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n].$$

Now define  $W_i(S_1, \dots, S_n) := v[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n](S_i)$ , for  $S_1, \dots, S_n \subset [m]$ . Then

$$\begin{aligned} W_i(S_1, \dots, S_n) &= 2^{-m} \sum_{R \subset [m]} T(R) \tilde{f}(S_1, \dots, S_{i-1}, S_i \triangle R, S_{i+1}, \dots, S_n) \\ &= 2^{-m} \sum_{\tilde{R} \subset [mn]} T_i(\tilde{R}) \tilde{f}((S_1, \dots, S_n) \triangle \tilde{R}) \end{aligned}$$

where for  $R_1, \dots, R_n \subset [m]$  we set  $T_i(R_1, \dots, R_n) = T(R_i)$  if  $R_j = \emptyset$  for all  $j \neq i$

and  $T_i(R_1, \dots, R_n) = 0$  otherwise. Thus, with convolutions of functions on  $\mathbb{Y}^n$  (or equivalently, on the power set of  $[nm]$ ) defined analogously to (2.2.9), we have

$$W_i = 2^{m(n-1)} T_i * \tilde{f}. \quad (2.2.14)$$

For  $F$  a real-valued function on  $\mathbb{Y}^n$  (or equivalently, on the power set of  $[mn]$ ), we define the Walsh-Fourier transform of  $F$  analogously to (2.2.10), by  $\hat{F}(S) = 2^{-nm} \sum_{B \subset [nm]} u_S(B) F(B)$  for  $S \subset [mn]$ . Writing  $S = (S_1, \dots, S_n)$  with  $S_1, \dots, S_n \subset [m]$ , and  $B = (B_1, \dots, B_n)$  similarly, we have  $u_S(B) = \prod_{j=1}^n u_{S_j}(B_j)$ . Hence

$$\begin{aligned} \hat{T}_i(S_1, \dots, S_n) &= 2^{-mn} \sum_{B=(B_1, \dots, B_n) \subset [mn]} T_i(B) u_{S_1}(B_1) \cdots u_{S_n}(B_n) \\ &= 2^{-mn} \sum_{B_i \subset [m]} T(B_i) u_{S_i}(B_i) \\ &= 2^{-mn+m} \hat{T}(S_i) = 2^{m(1-n)} |S_i|^{1/2}. \end{aligned}$$

Thus by (2.2.11) and (2.2.14),  $\hat{W}_i(S_1, \dots, S_n) = |S_i|^{1/2} \hat{f}(S_1, \dots, S_n)$ , so by Parseval's equation for functions on  $\mathbb{Y}^n$ ,

$$\begin{aligned} \|W_i\|_2^2 &= \sum_{S_1, \dots, S_n \subset [m]} (\hat{W}_i(S_1, \dots, S_n))^2 \\ &= \sum_{S_1, \dots, S_n \subset [m]} |S_i| \hat{f}(S_1, \dots, S_n)^2. \end{aligned} \quad (2.2.15)$$

But also,

$$\begin{aligned} \|W_i\|_2^2 &= 2^{-mn} \sum_{S_1, \dots, S_n} (v[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n](S_i))^2 \\ &= 2^{-mn} 2^m \sum_{S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n} \|v[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n]\|_2^2 \\ &= 2^{m(1-n)} \sum_{S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n} w(h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n])/4, \end{aligned}$$

where for the last line we have used (2.2.12). By (2.2.8),

$$w(h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n]) \leq 3k^2 q \log(4/q),$$

and also  $w(h[S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n]) = 0$  if  $\tilde{f}(S_1, \dots, S_{i-1}, \cdot, S_{i+1}, \dots, S_n)$  is a constant function. Hence,

$$\|W_i\|_2^2 \leq (3/4)k^2q \log(4/q)I_{f,\mathbf{p}}(i).$$

Summing over  $i$  and combining with (2.2.15), we obtain that

$$\sum_{\mathbf{S}=(S_1,\dots,S_n)} \hat{f}(\mathbf{S})^2 \|\mathbf{S}\| \leq (3/4)k^2q \log(4/q) \sum_{i=1}^k \delta_i,$$

where we set  $\delta_i := I_{f,\mathbf{p}}(i)$  and  $\|\mathbf{S}\| := \sum_{i=1}^k |S_i|$ .

Let  $\mathcal{S}_1 := \{\mathbf{S} : \|\mathbf{S}\| \geq 2k^2(t(1-t))^{-1}q \log(4/q) \sum_{i=1}^k \delta_i\}$ . Then

$$\begin{aligned} \sum_{\mathbf{S} \in \mathcal{S}_1} \hat{f}(\mathbf{S})^2 &\leq \frac{t(1-t) \sum_{\mathbf{S} \in \mathcal{S}_1} \|\mathbf{S}\| \hat{f}(\mathbf{S})^2}{2k^2q \log(4/q) \sum_{i=1}^k \delta_i} \\ &\leq \frac{3t(1-t)}{8}, \end{aligned} \tag{2.2.16}$$

whereas by Parseval's equation, since  $\hat{f}(\emptyset) = \mathbb{E}f(X) = h$  and  $f(\cdot) \in \{0, 1\}$ ,

$$\sum_{\{\mathbf{S} : \|\mathbf{S}\| > 0\}} \hat{f}(\mathbf{S})^2 = \|\tilde{f}\|_2^2 - (\mathbb{E}f(X))^2 = t(1-t). \tag{2.2.17}$$

Next, for  $i \in [n]$  we define the function  $R_i$  on  $\mathbb{Y}^n$  by

$$\begin{aligned} R_i &:= \sum_{S_1, \dots, S_n \subset [m] : S_i \neq \emptyset} \hat{f}(S_1, \dots, S_n) u_{S_1, \dots, S_n} \\ &= \tilde{f} - \sum_{S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n \subset [m]} \hat{f}(S_1, \dots, \emptyset, \dots, S_n) u_{S_1, \dots, \emptyset, \dots, S_n} \end{aligned}$$

where we have used the Walsh-Fourier expansion of  $\tilde{f}$ , and where it is to be understood that the  $\emptyset$  takes the place of  $S_i$  in the sequence  $(S_1, \dots, \emptyset, \dots, S_n)$ . Now,

$$\hat{f}(S_1, \dots, \emptyset, \dots, S_n) = \langle \tilde{f}, u_{(S_1, \dots, \emptyset, \dots, S_n)} \rangle$$

$$\begin{aligned}
&= 2^{-mn} \sum_{B_1, \dots, B_n \subset [m]} \tilde{f}(B_1, \dots, B_n) \prod_{j: j \neq i} u_{S_j}(B_j) \\
&= 2^{m(1-n)} \sum_{B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n \subset [m]} g_i(B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n) \prod_{j: j \neq i} u_{S_j}(B_j),
\end{aligned}$$

where we set  $g_i(B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n)$  to be the value of  $\tilde{f}(B_1, \dots, B_n)$  averaged over all values of  $B_i$ . Hence

$$\hat{\tilde{f}}(S_1, \dots, \emptyset, \dots, S_n) = \hat{g}_i(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n)$$

and so by a further Walsh-Fourier expansion, for any  $B_1, \dots, B_n \subset [m]$  we have

$$R_i(B_1, \dots, B_n) = \tilde{f}(B_1, \dots, B_n) - g_i(B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n).$$

Therefore  $|R_i(\mathbf{B})| \leq 1$  for all  $\mathbf{B} = (B_1, \dots, B_n) \subset [mn]$ , and  $R_i(\mathbf{B}) = 0$  whenever  $h[B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n]$ , defined by (2.2.13), is a constant function. Therefore, writing  $\|g\|_p$  for  $(2^{-mn} \sum_{\mathbf{B} \subset [mn]} |g(\mathbf{B})|^p)^{1/p}$  for any real-valued function  $g$  defined on  $\mathbb{Y}^n$  and any  $p \geq 1$ , we have that

$$\|R_i\|_{4/3}^{4/3} \leq I_{f, \mathbf{p}}(i),$$

and therefore by Lemma 4 of [7], for  $\varepsilon = 3^{-1/2}$ ,

$$\|T_\varepsilon R_i\|_2^2 \leq \|R_i\|_{1+\varepsilon^2}^2 \leq \delta_i^{3/2}, \quad (2.2.18)$$

where we set

$$T_\varepsilon R_i := \sum_{S \subset [mn]} \hat{R}_i(S) \varepsilon^{|S|} u_S.$$

Since  $\hat{R}_i(S_1, \dots, S_n)$  is zero or  $\hat{\tilde{f}}(S_1, \dots, S_n)$ , according to whether  $S_i$  is empty or not, so by Parseval's identity

$$\|T_\varepsilon R_i\|_2^2 = \sum_{\mathbf{S}=(S_1, \dots, S_n)} \hat{\tilde{f}}(\mathbf{S})^2 \varepsilon^{2\|\mathbf{S}\|} \mathbf{1}\{S_i \neq \emptyset\}. \quad (2.2.19)$$

For  $\mathbf{S} = (S_1, \dots, S_n) \subset [mn]$  let  $\mu(\mathbf{S})$  denote the number of  $i$  such that  $S_i \neq \emptyset$ .

Comparing (2.2.18) with (2.2.19) and summing over  $i$  yields

$$\sum_{\mathbf{S}} \hat{f}(\mathbf{S})^2 \varepsilon^{2\|\mathbf{S}\|} \mu(\mathbf{S}) \leq \sum_{i=1}^n \delta_i^{3/2}.$$

Let  $\mathcal{S}_2$  be the set of  $\mathbf{S}$  such that  $1 > \varepsilon^{2\|\mathbf{S}\|} \geq \left(2 \sum_i \delta_i^{3/2}\right) / (t(1-t))$ . Then

$$\begin{aligned} \sum_{\mathbf{S} \in \mathcal{S}_2} \hat{f}(\mathbf{S})^2 &\leq \sum_{\mathbf{S} \in \mathcal{S}_2} \frac{\mu(\mathbf{S}) \varepsilon^{2\|\mathbf{S}\|} \hat{f}(\mathbf{S})^2 t(1-t)}{2 \sum_i \delta_i^{3/2}} \\ &\leq t(1-t)/2. \end{aligned}$$

Combined with (2.2.16) and (2.2.17), since  $(3/8) + (1/2) < 1$ , this shows that there exists  $\mathbf{S}$  with  $\|\mathbf{S}\| > 0$  lying neither in  $\mathcal{S}_1$  nor in  $\mathcal{S}_2$ . Choosing such an  $\mathbf{S}$ , since  $\mathbf{S} \notin \mathcal{S}_2$  we have  $3^{-\|\mathbf{S}\|} < \left(2 \sum_i \delta_i^{3/2}\right) / (t(1-t))$  so that

$$\|\mathbf{S}\| > \log \left( \frac{t(1-t)}{2 \sum_i \delta_i^{3/2}} \right) / \log 3$$

but also  $\mathbf{S} \notin \mathcal{S}_1$ , so that

$$2k^2 q \log(4/q) \sum_i \delta_i > t(1-t) \log \left( \frac{t(1-t)}{2 \sum_i \delta_i^{3/2}} \right) / \log 3.$$

Suppose (2.2.6) holds. Then, setting  $\alpha := aq^2(\log(4/q))^2$  and  $I_f := \sum_{i=1}^n \delta_i$ , we have that  $\delta_i \leq \alpha$  for all  $i$  so that  $\sum_i \delta_i^{3/2} \leq \alpha^{1/2} I_f$ . Since  $2 \log 3 < 3$ , we have that

$$3k^2 I_f q \log(4/q) > t(1-t) \log \left( \frac{t(1-t)}{2\alpha^{1/2} I_f} \right).$$

Setting  $x := t(1-t)/(q \log(4/q))$  and  $b := I_f/x$ , we have

$$I_f > \left( \frac{x}{3k^2} \right) \log \left( \frac{t(1-t)}{2\alpha^{1/2}(q \log 4/q)bx} \right) = \left( \frac{x}{3k^2} \right) \log \left( \frac{1}{2\alpha^{1/2}b} \right).$$

Since  $I_f = bx$  it follows that  $b \geq (1/3)k^{-2} \log(1/(2\alpha^{1/2}b))$ , and therefore  $b + (1/3)k^{-2} \log b \geq (1/3)k^{-2} \log(1/(2\alpha^{1/2}))$ .

Since  $(\log u)/u \leq e^{-1}$  for all  $u > 0$ , and since we assume  $a \leq 1/16$  so that

$\log(a^{-1/2}) \geq 2 \log 2$ , therefore

$$2b \geq b + (1/3)k^{-2} \log b \geq (1/3)k^{-2} \log(1/(2a^{1/2})) \geq (1/6)k^{-2} \log a^{-1/2}.$$

Therefore  $b \geq (24k^2)^{-1} \log(1/a)$ , which implies (2.2.7).  $\square$

We now give the proof of Proposition 2.2.1, which is adapted from that of Lemma 1 in [6].

*Proof of Proposition 2.2.1.* Note that  $\gamma \leq \min(p_0, p_k + \gamma) \leq q_{\max}$ . Therefore by the assumption (2.2.1),  $\log m \geq 200(\log 2) \log(4/q_{\max})$ . Hence  $m \geq q_{\max}^{-9}$ , and also  $m \geq 16^4$ .

For  $0 \leq h \leq \gamma$  set  $\underline{\mathbf{r}}(h) = \underline{\mathbf{p}} + (-h, 0, \dots, 0, h)$ . Let  $g(h) = \mathbb{P}_{\underline{\mathbf{r}}(h)}^n(E)$ . By assumption,  $\underline{\mathbf{q}}$  dominates  $\underline{\mathbf{r}}(\gamma)$ . Therefore  $\mathbb{P}_{\underline{\mathbf{q}}}^n(E) \geq \mathbb{P}_{\underline{\mathbf{r}}(\gamma)}^n(E) = g(\gamma)$ . We shall use a form of the Margulis-Russo formula, namely

$$g'(h) = I_{f, \underline{\mathbf{r}}(h)} := \sum_{j=1}^n I_{f, \underline{\mathbf{r}}(h)}(j), \quad \forall h \in (0, \gamma), \quad (2.2.20)$$

where  $f$  is the indicator of event  $E$  and  $I_{f, \underline{\mathbf{p}}}(j)$  is the influence of the  $j$ th coordinate on the function  $f$ , as in Lemma 2.2.3. To see (2.2.20), for  $h_1, \dots, h_n \in (0, \gamma)$  let  $u(h_1, \dots, h_n)$  denote the probability of event  $E$  under the measure  $\prod_{i=1}^n \mathbb{P}_{\underline{\mathbf{r}}(h_i)}$  and for probability vectors  $\underline{\mathbf{p}}_1, \dots, \underline{\mathbf{p}}_n$  on  $\{0, 1, \dots, k\}$  and  $j \in [n]$  let  $I_{f, (\underline{\mathbf{p}}_1, \dots, \underline{\mathbf{p}}_n)}(j)$  denote the probability that the  $j$ th coordinate of  $X$  is pivotal for  $f$ , where  $X$  is a random element of  $\{0, 1, \dots, k\}^n$  with distribution  $\prod_{i=1}^n \underline{\mathbf{p}}_i$ . Then for  $j \in [n]$  and  $\varepsilon > 0$  with  $h_j + \varepsilon < \gamma$ , we can find coupled  $\{0, 1, \dots, k\}^n$ -valued random vectors  $X$  and  $X'$  with respective distributions  $\prod_{i=1}^n \mathbb{P}_{\underline{\mathbf{r}}(h_i)}$  and  $\prod_{i=1}^n \mathbb{P}_{\underline{\mathbf{r}}'_i}$ , where we set  $\underline{\mathbf{r}}'_i = \underline{\mathbf{r}}(h_i)$  except for  $i = j$ , and  $\underline{\mathbf{r}}'_j = \underline{\mathbf{r}}(h_j + \varepsilon)$ , and such that  $\mathbb{P}[X = X'] = 1 - \varepsilon$  and if  $X \neq X'$  then  $X_j = 0$  and  $X'_j = k$ , with  $X_i = X'_i$  for all  $i \neq j$ . Then  $f(X) \leq f(X')$ , with equality except when (i)  $X \neq X'$  and (ii) the  $j$ th coordinate of  $X$  is pivotal for  $f$ . Therefore

$$\mathbb{P}[f(X') \neq f(X)] = \varepsilon I_{f, (\underline{\mathbf{r}}(h_1), \dots, \underline{\mathbf{r}}(h_k))}(j)$$

so that  $\frac{\partial}{\partial h_j} u(h_1, \dots, h_n) = I_{f, (\underline{\mathbf{r}}(h_1), \dots, \underline{\mathbf{r}}(h_k))}$ , and then we obtain (2.2.20) by the chain rule, since  $g(h) = u(h, h, \dots, h)$ .

Next we show that for  $0 \leq h \leq \gamma$  we have

$$I_{f,\underline{r}(h)} \geq \frac{g(h)(1-g(h)) \log m}{96k^2 q_{\max} \log(4/q_{\max})}. \quad (2.2.21)$$

First suppose  $I_{f,\underline{r}(h)}(j) \geq m^{-1/2}$  for some  $j \in [n]$ . Then by the symmetry assumption we have  $I_{f,\underline{r}(h)}(j) \geq m^{-1/2}$  for at least  $m$  values of  $j$ , so that (using that  $m \geq q_{\max}^{-9}$ ) we have  $I_{f,\underline{r}(h)} \geq m^{1/2} \geq m^{1/3} q_{\max}^{-1.5}$ , and since  $\log m \leq 3m^{1/3}$ , this implies (2.2.21).

Now suppose instead that  $I_{f,\underline{r}(h)}(j) < m^{-1/2}$  for all  $j \in [n]$ . Then since  $m \geq q_{\max}^{-9}$  we have  $I_{f,\underline{r}(h)}(j) < m^{-1/4} q_{\max}^2$  for all  $j \in [n]$ . Setting

$$a := \max_{j \in [n]} I_{f,\underline{r}(h)}(j) / (q_{\max}^2 (\log(4/q_{\max}))^2),$$

we have that  $a \leq m^{-1/4} \leq 16$ ; also  $p_{\max}(\underline{r}(h)) \leq q_{\max}$ , so by Lemma 2.2.3 we have

$$I_{f,\underline{r}(h)} \geq \frac{g(h)(1-g(h)) \log(1/a)}{24k^2 q_{\max} \log(4/q_{\max})} \geq \frac{g(h)(1-g(h)) \log m}{96k^2 q_{\max} \log(4/q_{\max})}$$

which implies (2.2.21).

For  $0 \leq h \leq \gamma$  let  $\tilde{g}(h) = \log(g(h)/(1-g(h)))$ . By (2.2.20) and (2.2.21) we have

$$\frac{d\tilde{g}}{dh} = (g(1-g))^{-1} \frac{dg}{dh} \geq \frac{\log m}{96k^2 q_{\max} \log(4/q_{\max})}.$$

Since  $g(0) = \mathbb{P}_{\underline{p}}^n(E) \geq \eta$ , by assumption, we have  $\tilde{g}(0) \geq \log \eta = -\log(1/\eta)$ , and using the assumption (2.2.1), we obtain that

$$\tilde{g}(\gamma) \geq -\log(1/\eta) + \frac{\gamma \log m}{96k^2 q_{\max} \log(4/q_{\max})} \geq \log(1/\eta)$$

which implies  $g(\gamma) > 1 - \eta$ , and therefore also  $\mathbb{P}_{\underline{q}}^n(E) > 1 - \eta$ .  $\square$

## 2.3 Box crossings for eRSA

We now return to eRSA. As mentioned in Section 2.1, we shall seek to show that if the probability of crossing a  $3n \times n$  rectangle is non-vanishing for large  $n$ , then



after slightly increasing  $\lambda$  or  $p$  this probability becomes large. Ultimately, we shall do this using the sharp threshold result of Proposition 2.2.1 via a discretization of time, considering only arrival times that come before some suitably high cut off time. Before we are ready to do this, we need a number of results on box crossings and the effect of small changes in various parameters, so as to control the errors involved in the various approximations and discretizations we shall introduce.

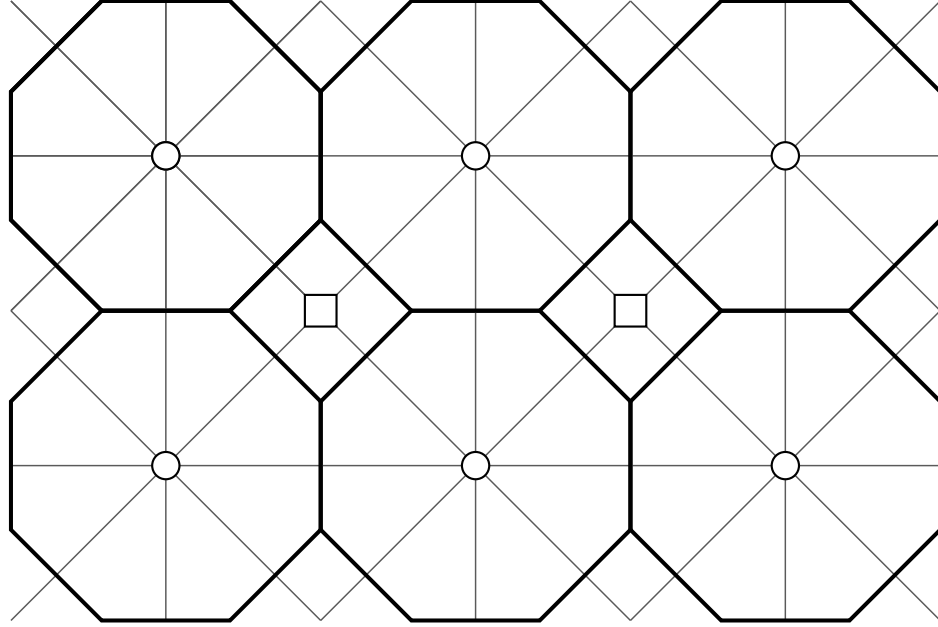


Figure 2-1: A section of the lattice used for enhanced RSA, with grey lines connecting adjacent nodes and black lines demonstrating the boundaries between adjacent shapes in the linked face percolation model.

From the enhanced RSA model of Section 2.1 construct a dependent face percolation model on a truncated square tiling as follows: colour the octagon centred at  $x \in \mathbb{Z}^2$  black if  $x$  is in the even phase, otherwise colouring it white. As in Section 1 we denote the diamond at the top right corner of the octagon centred at  $x$  by  $x'$ , and colour it black if the enhancement variable  $x'$  is in the even phase, and white otherwise.

We shall compensate the error due to time-discretization by introducing a time-*delay* at the even sites. We shall therefore consider a similar face percolation model, where the arrivals at the even sites have been slightly delayed.

Let  $\mathbb{P}_{\lambda,p,\delta}$  denote the probability measure for the enhanced RSA model where

the arrivals at the even sites are delayed by  $\delta$ . In a similar manner to [21] we can construct this measure from a collection of independent variables denoted  $T_x$  and  $T_{x'}$  where  $x \in \mathbb{Z}^2$ . Here  $T_x$  determines the arrival time  $t_x$  at site  $x$ , and  $T_{x'}$  is a uniform variable that determines whether the diamond site  $x'$  is black or white;  $T_x$ ,  $x \in \mathbb{Z}^2$  is exponentially distributed with parameter 1 for odd  $x$ , and with parameter  $\lambda$  for even  $x$ , while  $T_{x'}$  is a uniform(0,1) random variable. The arrival time  $t_x$  at  $x$  is  $t_x = T_x$  for odd  $x$  and is  $t_x = T_x + \delta$  for even  $x$ ,

Since the precise arrival times at the sites do not matter for the resulting distribution, merely the order of arrivals, we arrive at the same distribution if we move the arrival times at all octagon sites forward by amount  $\delta$ . Then by conditioning on the first arrival time at an odd site being at least  $\delta$  and using the memoryless property of the exponential distribution we can arrive at the equivalent distribution where the arrival time  $t_x$  at an even site is  $T_x$  and at an odd site is 0 with probability  $1 - e^{-\delta}$ , otherwise taking the value  $T_x$ . Then we can produce a distribution for  $\mathbb{P}_{\lambda,p,\delta}$  where we have  $T_x, x \in \mathbb{Z}^2$  with parameter 1 for odd  $x$  and  $\lambda$  for even  $x$ , along with uniform(0,1) random variables  $U_x, x \in \mathbb{Z}^2$  and  $T_{x'}, x \in \mathbb{Z}^2$ . For  $x \in \mathbb{Z}^2$  we now set the arrival time  $t_x$  to be 0 if  $x$  is odd and  $U_x \leq 1 - e^{-\delta}$ . We set  $t_x$  to be  $T_x$ , and set  $x'$  to be in the even phase if  $T_{x'} < p$  and in the odd phase otherwise.

In the dependent face percolation model given above with the states of sites distributed according to  $\mathbb{P}_{\lambda,p,\delta}$ , given  $\rho \in (0, \infty)$  and  $n \in \mathbb{N}$  with  $\rho n \geq 1$ , let  $H_{n,\rho}$  denote the event that there is a horizontal black crossing of the rectangle

$$R(2n, \rho) := [-\lfloor \rho n \rfloor, \lfloor \rho n \rfloor - 1] \times [-n, n - 1]$$

and set

$$h_\rho(n, \lambda, p, \delta) := \mathbb{P}_{\lambda,p,\delta}(H_{n,\rho}).$$

Next we require the concept of a site being *pivotal* for event  $H_{n,\rho}$ . The definition of this concept will depend on whether it is an odd site, an even site or a diamond site.

We shall say an odd site  $x$  is pivotal for the event  $H_{n,\rho}$  if  $H_{n,\rho}$  occurs when we set the arrival time  $t_x$  to  $T_x$  but if we were to change the arrival time  $t_x$  to 0 (leaving all other variables constant),  $H_{n,\rho}$  would no longer occur.

We shall say that an even site  $x$  is pivotal for event  $H_{n,\rho}$  if this event occurs

when the arrival time at the site  $x$  is  $T_x$ , but does not occur if we delay the arrival time at  $x$  by an independent exponential random variable with rate  $\lambda$  called  $T$ .

For  $y \in \mathbb{Z}^2$ , we say the diamond site  $y'$  is pivotal for the event  $H_{n,\rho}$  if  $H_{n,\rho}$  occurs when  $y'$  is black but does not occur when  $y'$  is white.

For any octagon or diamond site  $z$  we define

$$\phi_{\lambda,p,\delta,\rho}(n, z) := \mathbb{P}_{\lambda,p,\delta}[z \text{ is pivotal for event } H_{n,\rho}].$$

**Proposition 2.3.1.** *For any  $\lambda, p, n$  and  $\rho > 0$ , and for any  $\delta > 0$  it is the case that*

$$\frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial p} = \sum_{x \in \mathbb{Z}^2} \phi_{\lambda,p,\delta,\rho}(n, x') \quad (2.3.1)$$

and

$$\frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial \delta} = -e^{-\delta} \sum_{x \in \mathbb{Z}^2: x \text{ odd}} \phi_{\lambda,p,\delta,\rho}(n, x). \quad (2.3.2)$$

Also,

$$\frac{\partial h_\rho(n, \lambda, p, 0)}{\partial \lambda} = \lambda^{-1} \sum_{x \in \mathbb{Z}^2: x \text{ even}} \phi_{\lambda,p,0,\rho}(n, x). \quad (2.3.3)$$

For  $x, y \in \mathbb{Z}^2$ , we shall say site  $x$  *affects* site  $y$  if there is some self-avoiding path in  $\mathbb{Z}^2$  starting at a neighbour of  $x$  and ending at  $y$ , such that if the odd sites along this path are listed in order as  $x_1, x_2, \dots, x_m$ , then  $T_{x_1} \leq T_{x_2} \leq \dots \leq T_{x_m}$ . If a site  $x$  does not affect a site  $y$ , no change to the arrival time at  $x$  with all other arrival times remaining fixed can alter the state of  $y$ . By the union bound, if the graph distance between  $x$  and  $y$  in  $\mathbb{Z}^2$  is denoted  $d(x, y)$  and satisfies  $d(x, y) \geq 2$ , then

$$\mathbb{P}[y \text{ affects } x] \leq \frac{4^{d(x,y)}}{[d(x, y)/2]!}, \quad (2.3.4)$$

*Proof of Proposition 2.3.1.* Equations (2.3.1) and (2.3.3) are as in Proposition 4.1 of [21], and the proof there translates directly to this model.

For (2.3.2), fix  $n, \rho, p, \lambda, \delta$ . Enumerate the odd sites of  $\mathbb{Z}^2$  in some manner as  $x_1, x_2, \dots$ . Given  $k \in \mathbb{N}$  and  $\varepsilon > 0$ , let  $\mathbb{P}_{\delta,k,\delta+\varepsilon}$  denote probability for a model where  $t_{x_i} = 0$  if  $U_{x_i} \leq 1 - e^{-\delta}$  for  $i = 1, 2, \dots, k-1$  and  $t_{x_i} = 0$  if  $U_{x_i} \leq 1 - e^{-\delta-\varepsilon}$  for  $i = k, k+1, \dots$  and otherwise  $t_{x_i} = T_{x_i}$  with  $T_{x_i}$  independent exponential random variables with parameter 1, enhancement parameter  $p$ , and arrival times

at the even sites independent exponential variables with parameter  $\lambda$ .

Let  $A(x)$  be the event that the site  $x$  affects some site in  $R(2n, \rho)$ . Then

$$\begin{aligned} 0 \leq h_\rho(n, \lambda, p, \delta) - \mathbb{P}_{\delta, k, \delta + \varepsilon}[H_{n, \rho}] &\leq \mathbb{P}_{\delta, k, \delta + \varepsilon}[\cup_{j=k}^{\infty} A(x_j)] \\ &\rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

by (2.3.4). Thus,

$$\begin{aligned} h_\rho(n, \lambda, p, \delta + \varepsilon) - h_\rho(n, \lambda, p, \delta) &= \mathbb{P}_{\delta, 1, \delta + \varepsilon}[H_{n, \rho}] - \lim_{k \rightarrow \infty} \mathbb{P}_{\delta, k, \delta + \varepsilon}[H_{n, \rho}] \\ &= \sum_{k=1}^{\infty} (\mathbb{P}_{\delta, k, \delta + \varepsilon}[H_{n, \rho}] - \mathbb{P}_{\delta, k+1, \delta + \varepsilon}[H_{n, \rho}]). \end{aligned}$$

Given  $\delta' > 0$  we now define  $F_k(\delta, \delta')$  to be the event that  $H_{n, \rho}$  occurs if we take the arrival time at site  $x_k$  to be  $T_{x_k}$ , but not if we take it to be 0, where an odd site  $x_j$  has arrival time 0 with probability  $1 - e^{-\delta}$  if  $j < k$  and with probability  $1 - e^{-\delta'}$  if  $j > k$ , otherwise having as arrival time  $T_{x_j}$  (the dependence of  $F$  on  $p, \lambda$  and  $n$  is suppressed). With the variables as described above, we see that

$$\mathbb{P}_{\delta, k, \delta + \varepsilon}[H_{n, \rho}] - \mathbb{P}_{\delta, k+1, \delta + \varepsilon}[H_{n, \rho}] = -e^{-\delta}(1 - e^{-\varepsilon})\mathbb{P}[F_k(\delta, \delta + \varepsilon)]. \quad (2.3.5)$$

Couple  $F_k(\delta, \delta + \varepsilon)$  and  $F_k(\delta, \delta)$  by fixing the collection of random variables  $U_{x_j}$  for  $j \in \mathbb{N}$ . For  $K \in \mathbb{N}$  let  $B(2K + 1) := [-K, K] \times [-K, K]$ . Then for any integer  $K > 3n$  we see that

$$\begin{aligned} F_k(\delta, \delta + \varepsilon) \triangle F_k(\delta, \delta) &\subset (\cup_{x \in \mathbb{Z}^2 \setminus B(2K+1)} A(x)) \cup \\ &\cup (\cup_{\{j > k: x_j \in B(2K+1)\}} \{1 - e^{-\delta} < U_{x_j} < 1 - e^{-(\delta + \varepsilon)}\}). \end{aligned}$$

For any fixed  $K$ , the probability of the event

$$\cup_{\{j > k: x_j \in B(2K+1)\}} \{1 - e^{-\delta} < U_{x_j} < 1 - e^{-(\delta + \varepsilon)}\}$$

vanishes as  $\varepsilon \downarrow 0$ , and the probability of the event  $\cup_{x \in \mathbb{Z}^2 \setminus B(2K+1)} A(x)$  is indepen-

dent of  $\varepsilon$  and vanishes as  $K \rightarrow \infty$ . Then (2.3.5) yields

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (\mathbb{P}_{\delta,k,\delta+\varepsilon} [H_{n,\rho}] - \mathbb{P}_{\delta,k+1,\delta+\varepsilon} [H_{n,\rho}]) &= -e^{-\delta} \mathbb{P}[F_k(\delta, \delta)] \\ &= -e^{-\delta} \phi_{\lambda,p,\delta,\rho}(n, x_k). \end{aligned} \quad (2.3.6)$$

Finally note that  $\mathbb{P}[F_k(\delta, \delta + \varepsilon)]$  is bounded by  $\mathbb{P}[A(x_k)]$ , which is independent of  $\varepsilon$  and summable in  $k$  by (2.3.4), so by (2.3.5), (2.3.6), and the dominated convergence theorem we have

$$\frac{\partial^+ h_\rho}{\partial \delta} = \lim_{\varepsilon \downarrow 0} \frac{h_\rho(n, \lambda, p, \delta + \varepsilon) - h_\rho(n, \lambda, p, \delta)}{\varepsilon} = -e^{-\delta} \sum_{k=1}^{\infty} \phi_{\lambda,p,\delta,\rho}(n, x_k).$$

A similar argument can be used to produce the same expression for the left partial derivative.  $\square$

Note that for any  $q > 0$  and fixed  $n$  we can find a distance  $r$  such that the probability that there exists a site at distance more than  $r$  from  $R(2n, \rho)$  that affects some site within  $R(2n, \rho)$  is less than  $q$ . Hence for  $\delta$  sufficiently small we have

$$\mathbb{P}_{\lambda,p,0}(H_{n,\rho}) - \mathbb{P}_{\lambda,p,\delta}(H_{n,\rho}) \leq q + (1 - e^{-\delta})(2\rho n + 2r)^2 \leq 2q,$$

and thus  $\mathbb{P}_{\lambda,p,\delta}(H_{n,\rho})$  is right continuous in  $\delta$  at  $\delta = 0$ .

We seek to bound the effect of a slight change in  $\delta$  in terms of the effect of a change in  $p$ . To do this we shall use a variant of arguments from [21]. Let  $y$  be an odd site and let  $r \in \mathbb{N}$ ; then define  $C_r = C_r(y)$  be the square of side length  $2r + 1$  centred at  $y$ . Define  $E_\rho(n, y, r)$  to be the event that if we use  $t_y = 0$  then (i) event  $H_{n,\rho}$  occurs if we change the colour of all sites in  $C_r$  to black (and leaving other sites unchanged) and (ii) event  $H_{n,\rho}$  does not occur if we change the colour of all sites in  $C_r$  to white.

**Lemma 2.3.1.** *Let  $\varepsilon \in (0, 1/2)$ . There exists a constant  $c_2 > 0$  such that for any odd  $y \in \mathbb{Z}^2$ , any  $n \in \mathbb{N}$ , any  $\rho \in \mathbb{N}$ , and*

$$(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times [0, 1]$$

we have

$$\phi_{\lambda,p,\delta,\rho}(n, y) \leq \mathbb{P}_{\lambda,p,\delta}[E_\rho(n, y, 1)] + \sum_{r=1}^{\infty} \frac{c_2^r \mathbb{P}_{\lambda,p,\delta}[E_\rho(n, y, r+1)]}{[r/2]}.$$

*Proof.* The arguments used to prove Lemma 5.2 of [21] apply equally here, once the necessary changes to notation have been made.  $\square$

For  $y \in \mathbb{Z}^2$ , let  $z_\rho(n, y)$  be the nearest even site in  $R(2(n-3), \rho)$  to  $y$  using Euclidean distance, taking the first according to the lexicographic ordering when there is a choice. Let  $z'_\rho(n, y) := z_\rho(n, y) + (1/2, 1/2)$ .

**Lemma 2.3.2.** *Let  $\varepsilon \in (0, 1/2)$ . There exists a constant  $c_3 \in (0, \infty)$  such that for any odd  $y \in \mathbb{Z}^2$ , any  $n \in \mathbb{N}$  with  $n \geq 60$ , any  $\rho \in \mathbb{N}$ , and any  $(\lambda, p, \delta, r) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1-\varepsilon] \times [0, 1] \times \mathbb{N}$ , we have that*

$$\mathbb{P}_{\lambda,p,\delta}[E_\rho(n, y, r)] \leq c_3^r \phi_{\lambda,p,\delta,\rho}(n, z'_\rho(n, y)) \mathbf{1}_{R(2(n+r), \rho)}(y).$$

*Proof.* The arguments used to prove Proposition 5.1 of [21] apply equally here, after necessary changes to notation.  $\square$

**Lemma 2.3.3.** *Let  $\varepsilon \in (0, 1/2)$ . There exists a constant  $c_4 = c_4(\varepsilon) \in (0, \infty)$  such that for any odd  $y$ ,  $n \in \mathbb{N}$ , any  $\rho \in \mathbb{N}$ , and  $(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1-\varepsilon] \times [0, 1]$ , we have*

$$\sum_{y \in \mathbb{Z}^2: y \text{ odd}} \phi_{\lambda,p,\delta,\rho}(n, y) \leq c_4 \sum_{z \in R(2(n-3), \rho): z \text{ even}} \phi_{\lambda,p,\delta,\rho}(n, z').$$

*Proof.* Using our Lemma 2.3.2, the proof is as in the first step of the proof of Proposition 3.1 in [21].  $\square$

**Corollary 2.3.1.** *Let  $\varepsilon \in (0, 1/2)$ . There exists a constant  $c_5 = c_5(\varepsilon) \in (0, \infty)$  such that  $y, n \in \mathbb{N}$ , any  $\rho \in \mathbb{N}$ , and  $(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1-\varepsilon] \times [0, 1]$ , we have*

$$\left| \frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial \delta} \right| \leq c_5 \frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial p}. \quad (2.3.7)$$

*Proof.* The result follows immediately from Lemma 2.3.3 and Proposition 2.3.1.  $\square$

Now write  $h_\rho(n, \lambda, p)$  for  $h_\rho(n, \lambda, p, 0)$ . Also, define  $h'_\rho(n, \lambda, p)$  similarly but in terms of a white crossing. That is,  $h'_\rho(n, \lambda, p)$  denotes the probability that there is a horizontal white crossing of an arbitrary fixed  $2\lfloor \rho n \rfloor$  by  $2n$  rectangle in an eRSA model with parameters  $\lambda$  and  $p$ .

**Lemma 2.3.4.** *Let  $\lambda > 0$ ,  $p > 0$ ,  $\rho > 0$  be fixed. If  $\limsup_{n \rightarrow \infty} h_\rho(n, \lambda, p) > 0$  then  $\limsup_{n \rightarrow \infty} h_{\rho'}(n, \lambda, p) > 0$  for all  $\rho' > 0$ . If  $\limsup_{n \rightarrow \infty} h'_\rho(n, \lambda, p) > 0$  then  $\limsup_{n \rightarrow \infty} h'_{\rho'}(n, \lambda, p) > 0$  for all  $\rho' > 0$ .*

*Proof.* A weaker version of this result (with  $\liminf$  rather than  $\limsup$  in the hypothesis, and with  $\rho = 1$ ) is given by the proof of Proposition 3.2 in [21], based on that of Theorem 4.1 of [2]. The details on how to convert the proof to the stronger statement given here can be found in [1, Section 4].  $\square$

Note that by symmetry,  $h_1(n, 1, 1/2) = 1/2$ , and therefore we have for all  $\rho > 1$  that  $\limsup_{n \rightarrow \infty} h_\rho(n, 1, 1/2) > 0$ .

Finally, in order to produce a symmetric event on a finite space we require an approximation result for considering enhanced RSA on a discrete even torus. We shall be using Lemma 3.3 of [21], stated below. For a rectangle  $R = [a, b] \times [c, d]$ , we define  $E_{\text{dense}}(R, r)$  to be the event that no site in  $R$  is affected by any site outside  $[a - r, b + r] \times [c - r, d + r]$ .

**Lemma 2.3.5.** *Let  $\lambda > 0$ ,  $\rho \geq 1$ . Given  $s > 0$ , let  $R_s = [1, \lfloor s \rfloor] \times [1, \lfloor \rho s \rfloor]$ . Then  $\mathbb{P}_\lambda[E_{\text{dense}}(R_s, 2\lfloor s^{1/2} \rfloor)] \rightarrow 1$  as  $s \rightarrow \infty$ . Moreover,  $E_{\text{dense}}(R, r)$  depends only on arrival times within the larger rectangle.*

By  $\mathbb{T}(2n)$  we shall denote the torus formed from a  $2n$  by  $2n$  square of octagon sites and the diamond sites at the upper right corner of each octagon site. We shall arbitrarily choose an octagon site in the torus to be the origin, and from this we can define even and odd sites on  $\mathbb{T}(2n)$  and hence have enhanced RSA as before on the torus. Where required, we shall denote by  $P_{\lambda, p}^{\mathbb{T}(2n)}$  and  $P_{\lambda, p}^\Lambda$  the probability measures for enhanced RSA with parameters  $\lambda$  and  $p$  on the torus  $\mathbb{T}(2n)$  and on the full enhanced integer lattice  $\Lambda$  respectively.

**Lemma 2.3.6.** *Let  $n \in \mathbb{N}$ ,  $\lambda > 0$  and  $p \in (0, 1)$ , and let  $R$  be a rectangle with long side length at most  $2n - 4\sqrt{2n}$ . Then*

$$\left| P_{\lambda, p}^{\mathbb{T}(2n)}[H(R)] - P_{\lambda, p}^\Lambda[H(R)] \right| < e(n)$$

where  $H(R)$  is the event that  $R$  has a horizontal black crossing and  $e(n)$  is some  $o(1)$  function independent of  $R$ .

*Proof.* We can couple enhanced RSA on  $\Lambda$  and on  $\mathbb{T}(2n)$  such that the arrival times at integer sites and colours of diamond sites agree on  $\{(a, b) : 0 \leq a, b \leq 2n\}$ . By Lemma 2.3.5 the probability that there is a site within a rectangle contained within  $\{(a, b) : 2 \lceil \sqrt{2n} \rceil \leq a, b \leq 2n - 2 \lceil \sqrt{2n} \rceil\}$  whose colour disagrees with the colour of the associated site in  $\Lambda$  tends to 0 as  $n \rightarrow \infty$ , and so the result follows.  $\square$

For  $n \in \mathbb{N}$ ,  $X$  a graph on which an enhanced RSA model is defined, and  $R$  a collection of sites of  $X$ , let  $E_{\text{fast}}^n(R)$  be the event that for all  $z \in R \cap \mathbb{Z}^2$  the first arrival time at  $z$  is less than  $\sqrt{n}$ .

**Lemma 2.3.7.** *Let  $\lambda > 0$ ,  $p \in (0, 1)$ ,  $K \in \mathbb{R}_+$ . For any sequence of regions  $R_n$  where the number of octagon sites contained in  $R_n$  is  $O(n^K)$ , we have that*

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\lambda, p}(E_{\text{fast}}^n(R_n)) = 1.$$

*Proof.* This follows from the union bound and the exponential decay of the tail of the exponential distribution.  $\square$

## 2.4 Two key steps

In this section we present two results (Propositions 2.4.1 and 2.4.2) which play a key role in preparing for the proof of Theorem 2.1.1. The first of these propositions shows that the effect on the crossing probability  $h_\rho(n, \lambda, p, \delta)$  of a small change in  $\lambda$ , is comparable to the effect of a small change in  $p$ .

To prove this, we need to find an appropriate inequality connecting even sites being pivotal, and diamond sites being pivotal. Figure 2-2 demonstrates one of the four possible arrangements of occupied sites closest to the diamond site in question (the other possibilities being the reflection of the occupation locations and colour inversions of these two). In order for this diamond site to be pivotal, in addition to the sites locally having an arrangement of this form we also require that there is a black path from the left edge of the rectangle to one of the occupied black sites close to the diamond site, a black path from the right edge to the other



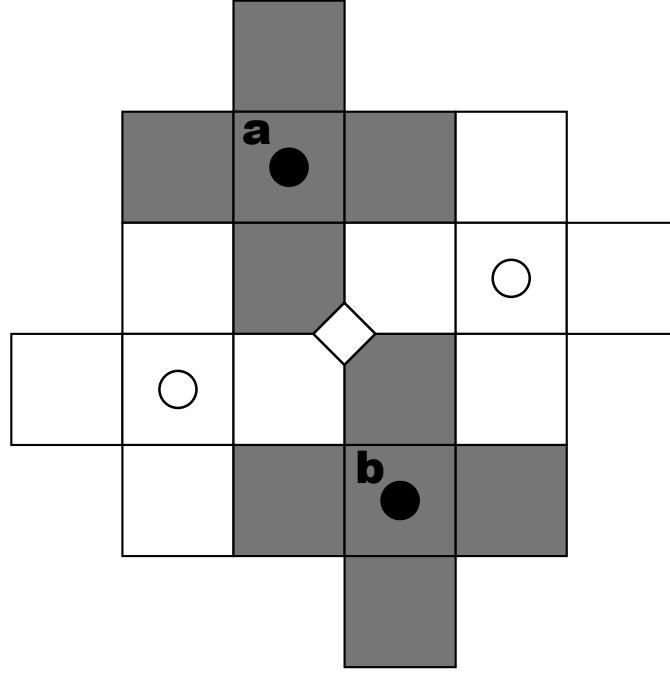


Figure 2-2: An example of the possible local arrangements of occupied and blocked sites such that a diamond site may be pivotal.

occupied black site, a white path from the top edge to one of the occupied white sites, and a white path from the bottom edge to the other occupied white site.

Recall the definition that for any octagon site  $x$ , the site  $x'$  is the site  $x + (1/2, 1/2)$ , and define similarly  $x''$  as the site  $x + (1/2, -1/2)$ .

**Lemma 2.4.1.** *For any  $\varepsilon \in (0, 1)$  there is a constant  $c_6 = c_6(\varepsilon) > 0$ , such that for any  $\lambda \in [\varepsilon, 1/\varepsilon]$ ,  $p \in [0, 1]$ ,  $\rho > 0$ ,  $n \in \mathbb{N}$ , and any even  $x \in \mathbb{Z}^2$  we have*

$$\phi_{\lambda,p,0,\rho}(n, x') \leq c_6 \phi_{\lambda,p,0,\rho}(n, x), \quad (2.4.1)$$

and

$$\phi_{\lambda,p,0,\rho}(n, x'') \leq c_6 \phi_{\lambda,p,0,\rho}(n, x). \quad (2.4.2)$$

*Proof.* We fix some chosen even site  $y$ , and we let  $\mathcal{S}_y = (S_x)_{x \in \Lambda}$  be the collection of arrival times and enhancement variables in one enhanced RSA process. In a similar manner to the proof of Proposition 5.1 of [21], we shall construct a coupled process  $\mathcal{U}_y = (U_x)_{x \in \Lambda}$ . For  $n \in \mathbb{N}$ , let  $B(2n+1)$  be the collection of octagon and diamond sites within  $[-n, n] \times [-n, n]$ . Let  $\mathcal{S}_y$  be as above, let  $\mathcal{T}_y = (T_x)_{x \in \Lambda}$

be the set of arrival times and enhancement variables in an independent RSA process, and let  $\mathcal{B} = (B_x)_{x \in \Lambda}$  be a collection of independent Bernoulli random variables with parameter 0.5. Then we define

$$\begin{aligned} U_x &= S_x & x \in \mathbb{Z}^2 \setminus (B(13) + y) \\ &= T_x & x \in \mathbb{Z}^2 \cap (B(7) + y) \\ &= B_x T_x + (1 - B_x) S_x & x \in \mathbb{Z}^2 \cap ((B(13) \setminus B(7)) + y) \\ &= S_x & x - (1/2, 1/2) \in \mathbb{Z}^2. \end{aligned}$$

We also define an independent exponential random variable  $T$  with parameter  $\lambda$ .

We now define events  $E_1$ ,  $E_2$  and  $E_3$  such that if all three events hold, then the site  $y$  is pivotal in the  $\mathcal{U}_y$  process.  $E_1$  is the event that the diamond site  $y'$  is pivotal for the  $\mathcal{S}_y$  process. We define  $E_2$  as a slightly altered version of the event  $E_1^{(3,6)}$  used in Lemma 5.1 of [21]; letting  $A_y(n, m)$  be the square annulus  $y + B(n) \setminus B(m)$ , we define  $E_2$  as

$$\begin{aligned} E_2 = & \bigcap \bigcap_{\{x \in A_y(13,7) \cap \mathbb{Z}^2 : S_x > 1 \text{ and } x \text{ is occupied in } \mathcal{S}_y\}} \{B_x = 1 \text{ and } T_x < 1\} \\ & \bigcap_{\{x \in A_y(13,7) \cap \mathbb{Z}^2 : S_x < 1 \text{ and } x \text{ is unoccupied in } \mathcal{S}_y\}} \{B_x = 1 \text{ and } T_x > 1\} \\ & \bigcap_{\{x \in A_y(13,7) \cap \mathbb{Z}^2 : S_x < 1 \text{ and } x \text{ is occupied in } \mathcal{S}_y\}} \{B_x = 0\} \\ & \bigcap_{\{x \in A_y(13,7) \cap \mathbb{Z}^2 : S_x > 1 \text{ and } x \text{ is unoccupied in } \mathcal{S}_y\}} \{B_x = 0\} \\ & \bigcap_{\{x \in A_y(7,3) \cap \mathbb{Z}^2 : x \text{ is occupied in } \mathcal{S}_y\}} \{0.5 < T_x < 1\} \\ & \bigcap_{\{x \in A_y(7,3) \cap \mathbb{Z}^2 : x \text{ is unoccupied in } \mathcal{S}_y\}} \{T_x > 1\}. \end{aligned}$$

Finally we define  $E_3$  as

$$\begin{aligned} E_3 := & \{T_y \leq 0.1\} \cap_{z \in y+B(3): z \text{ odd}} \{0.1 < T_z \leq 0.2\} \cap \\ & \cap_{z \in y+B(3): z \text{ even}, z \neq y} \{0.2 < T_z \leq 0.3\} \cap \{T > 0.2\}. \end{aligned}$$

We now consider the state of the  $\mathcal{U}_y$  process if all of these events occur. If the events  $E_2$  and  $E_3$  both hold, then every even octagon site within the square  $y + B(3)$  is occupied if we have the arrival time at  $y$  being  $T_y$ , but unoccupied if we delay the arrival at  $y$  by  $T$ . As noted in Lemma 5.1 of [21], provided  $E_2$  occurs then the states of sites outside  $y + B(7)$  in the  $\mathcal{U}_y$  process match the states

of those sites in the  $\mathcal{S}_y$  process. Now we consider any even octagon site within  $A_y(7, 3)$ . If this site was black in the  $\mathcal{S}_y$  process, then in the  $\mathcal{U}_y$  process it has arrival time less than 1 and any adjacent sites outside  $y + B(3)$  have arrival times at least 1, thus are unable to block it. Since all odd sites within  $y + B(3)$  are blocked by the arrival at  $y$ , it follows that the site under consideration has first arrival time strictly lower than all adjacent unblocked sites and hence is occupied.

Suppose  $y'$  is pivotal in  $\mathcal{S}_y$ . Without loss of generality, we assume that in the  $\mathcal{S}_y$  process the local arrangement of occupied sites at  $y'$  matches that in figure 2-2 and that the site labelled  $a$  has a black path connecting it to the left side of the rectangle, and that the site labelled  $b$  has a black path connecting it to the right side of the rectangle. By our argument and due to black paths being increasing in black sites, it follows that in the  $\mathcal{U}_y$  process there is a black path from the left side of the rectangle to  $a$ , from the site  $a$  to the site  $b$  due to all the sites in the square  $y + B(3)$  being black, and from the site  $b$  to the right side of the rectangle. As such, we see that in the  $\mathcal{U}_y$  process, if the events  $E_1$ ,  $E_2$  and  $E_3$  hold and we take  $T_y$  as the arrival time at  $y$  we have a horizontal black crossing of the rectangle.

A similar argument shows that if we delay the arrival at  $y$  by the random variable  $T$  and the events  $E_1$ ,  $E_2$  and  $E_3$  hold then we have a vertical white crossing of the rectangle, and thus the site  $y$  is pivotal. We then obtain (2.4.1) by noting that the events  $E_1$ ,  $E_2$ , and  $E_3$  are independent, that the probability of  $E_1$  is  $\phi_{\lambda,p,0,\rho}(n, y')$ , and that there is a strictly positive lower bound on the probabilities of the events  $E_2$  and  $E_3$ , uniformly over  $0 \leq p \leq 1$  and  $\varepsilon \leq \lambda \leq 1/\varepsilon$ .

A similar argument provides the second inequality (2.4.2), completing the proof.  $\square$

We are now able to give the first key result in this section.

**Proposition 2.4.1.** *Let  $\varepsilon \in (0, 1/2)$ . Then there is a constant  $c_7 = c_7(\varepsilon) \in (0, \infty)$  such that for any  $(\lambda, p, n) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times \mathbb{N}$  we have*

$$c_7^{-1} \frac{\partial h_3(n, \lambda, p)}{\partial \lambda} \leq \frac{\partial h_3(n, \lambda, p)}{\partial p} \leq c_7 \frac{\partial h_3(n, \lambda, p)}{\partial \lambda}, \quad (2.4.3)$$

*and moreover the second inequality of (2.4.3) holds for any  $(\lambda, p, n) \in [\varepsilon, 1/\varepsilon] \times [0, 1] \times \mathbb{N}$ .*

*Proof.* The second inequality of (2.4.3) follows immediately from Lemma 2.4.1, equation (2.3.3) and equation (2.3.1).

The first inequality of (2.4.3) is obtained as in the proof of Proposition 3.1 of [21].  $\square$

The second key result of this section (Proposition 2.4.2) says that if (at some  $(\lambda, p)$ ) we have non-vanishing probability of crossing a large rectangle of fixed aspect ratio, then after a slight increase of either  $\lambda$  or  $p$  we have probability close to 1 of crossing a rectangle of aspect ratio 3 the long way. We first prove this to be the case after a slight increase of *both*  $\lambda$  and  $p$ . It is in the proof of this that we use our sharp thresholds result.

**Lemma 2.4.2.** *Let  $\lambda > 0$  and  $p \in (0, 1)$ . Suppose there exists  $\rho > 0$  with  $\limsup_{n \rightarrow \infty} h_\rho(n, \lambda, p) > 0$ . Then for any  $\varepsilon \in (0, \min(p, 1 - p))$ , we have*

$$\limsup_{n \rightarrow \infty} h_3(n, \lambda(1 + \varepsilon), p + \varepsilon) > 1 - \varepsilon. \quad (2.4.4)$$

*Proof.* Set  $p^* = \limsup_{n \rightarrow \infty} h_9(n, \lambda, p)$ . By Lemma 2.3.4 we have  $p^* > 0$ .

We now consider a rectangle  $R_1 = R_1(n) = [1, 18n] \times [1, 2n]$  as a rectangle in the torus  $\mathbb{T} = \mathbb{T}(20n)$ . Let  $\delta = (\log n)^{-1/2}$  with  $n$  high enough that  $\varepsilon/4 > c_5\delta$  where  $c_5$  is the constant arising in Corollary 2.3.1. Let  $R_2$  be the  $20n$  by  $20n$  region in  $\Lambda$  which is identified with  $\mathbb{T}$ , let  $E_1$  be the event that  $R_1$  has a horizontal black crossing after the arrival times at all even sites are delayed by  $2\delta$ , and let  $E_{\text{dense}}^n$  be the event  $E_{\text{dense}}(R_1, 2 \lceil \sqrt{2n} \rceil)$ . Also let  $R_2^e$  be the set of even sites in  $R_2$  and let  $\mathbb{T}^e$  be the set of even sites in  $\mathbb{T}$ . Then using Lemmas 2.3.5 and 2.3.7, we have

$$\begin{aligned} \mathbb{P}_{\lambda, p+\varepsilon/2}^{\mathbb{T}} [E_1 \cap E_{\text{fast}}^n(\mathbb{T}^e)] &\geq \mathbb{P}_{\lambda, p+\varepsilon/2}^{\Lambda} [E_1 \cap E_{\text{dense}}^n \cap E_{\text{fast}}^n(R_2^e)] \\ &\geq \mathbb{P}_{\lambda, p+\varepsilon/2}^{\Lambda} (E_1) + o(1). \end{aligned}$$

Then noting that  $\mathbb{P}_{\lambda, p+\varepsilon/2}^{\Lambda} (E_1) = h_9(n, \lambda, p + \varepsilon/2, 2\delta)$  and that  $\mathbb{P}_{\lambda, p}^{\Lambda} (H_{n,9}) = h_9(n, \lambda, p, 0)$ , using (2.3.7) and the Mean Value Theorem, for infinitely many

$n$  we obtain

$$\begin{aligned}\mathbb{P}_{\lambda, p+\varepsilon/2}^{\mathbb{T}}[E_1 \cap E_{\text{fast}}^n(\mathbb{T}^e)] &\geq \mathbb{P}_{\lambda, p}^{\Lambda}(H_{n,9}) + o(1) \\ &\geq p^*/2.\end{aligned}\tag{2.4.5}$$

We seek to apply Proposition 2.2.1, which requires us to be working on an event of some order of symmetry in a discrete product space. Let  $E_2$  be the event that there is some  $18n$  by  $2n$  rectangle in  $\mathbb{T}$  with a horizontal black crossing after the arrival times at all even sites are delayed by  $2\delta$  and that also  $E_{\text{fast}}^n(\mathbb{T}^e)$  occurs. Clearly  $E_2$  is symmetric of order  $200n^2$  in permutations of sites by translations of the torus (modulo  $20n$ ) that send even sites to even sites, and  $\mathbb{P}_{\lambda, p+\varepsilon/2}^{\mathbb{T}}(E_2) \geq \mathbb{P}_{\lambda, p+\varepsilon/2}^{\mathbb{T}}[E_1 \cap E_{\text{fast}}^n(\mathbb{T}^e)]$ .

Given  $(\lambda_0, \lambda_1, \tilde{p}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times (0, 1)$  we construct a discrete version of enhanced RSA on the torus, where even sites have arrivals at rate  $\lambda_0$ , odd sites have arrivals at rate  $\lambda_1$  and diamond sites are black with probability  $\tilde{p}$ . At each site  $x \in \mathbb{Z}^2$  we shall divide the time axis up into blocks of length  $\delta$ , and discarding all blocks that had their start time later than  $n$  we have a product space  $\mathbb{T} \times \{-1, 0, 1, 2, \dots, \lfloor n/\delta \rfloor\}$  where  $(x, -1)$  represents the diamond site  $x' := x + (1/2, 1/2)$ , and  $(x, k)$  for  $k \in \{0, 1, \dots, \lfloor n/\delta \rfloor\}$  represents the site  $x$  at times in the interval  $I_\delta(k) := [k\delta, (k+1)\delta)$ . We denote the probability measure on this new space by  $\mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}$ .

We shall now construct a random field

$$\mathbf{X} = (X(x, k) : (x, k) \in \mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\})$$

with each  $X(x, k)$  taking values in  $\{0, 1, 2, 3\}$ . For an even site  $x$  and for  $k \geq 0$ , we set  $X(x, k) = 3$  if there is an attempted arrival at  $x$  within  $I_\delta(k)$  and  $X(x, k) \in \{0, 1, 2\}$  if there is no attempted arrival in this interval. For an odd site  $x$  and for  $k \geq 0$ , we set  $X(x, k) = 0$  if there is an attempted arrival at  $x$  within  $I_\delta(k)$  and  $X(x, k) \in \{1, 2, 3\}$  if there is no attempted arrival in this interval. For any site  $x$ , we put  $X(x, -1) \in \{2, 3\}$  if  $x'$  is black, and  $X(x, -1) \in \{0, 1\}$  if  $x'$  is white. To construct a representation of this model in a discrete product space we consider all arrivals at a site instead of solely the first, so that  $X(x, k_1)$  is independent of  $X(x, k_2)$  whenever  $k_1 \neq k_2$ . Where there is a choice of the value of  $X(z)$  for

$z \in \mathbb{T} \times \{-1, 0, 1, 2, \dots, \lfloor n/\delta \rfloor\}$  we choose randomly and independently of  $X(z')$  for all  $z' \neq z$  so that the distribution of  $\mathbf{X}$ , denoted  $\mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}$ , satisfies

$$\begin{aligned} \mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(X(z) = 3) &= 1 - e^{-\lambda_0 \delta}; \\ \mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(X(z) = 2) &= \tilde{p} + e^{-\lambda_0 \delta} - 1; \\ \mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(X(z) = 1) &= e^{-\lambda_1 \delta} - \tilde{p}; \\ \mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(X(z) = 0) &= 1 - e^{-\lambda_1 \delta}. \end{aligned} \tag{2.4.6}$$

Since we assume  $\tilde{p} \in (0, 1)$ , for large enough  $n$  these really are probabilities.

Let  $\mathbb{P}_{\lambda, p, \lambda'}^{\mathbb{T}}$  be the probability measure associated with the enhanced RSA model on the torus  $\mathbb{T}$  in which arrivals at even sites come at rate  $\lambda$  and at odd sites at rate  $\lambda'$ . Now let  $E_2^{\text{crude}}$  be the event that the state of  $\mathbf{X} := \{X(x, k) : (x, k) \in \mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}\}$  is such that  $E_2$  is possible given  $\mathbf{X}$ ; this can be seen as either an event on the discrete time torus  $\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}$  representing that there is some collection of arrival times at sites in the continuous time torus which satisfies  $E_2$  and is consistent with  $\mathbf{X}$ , or as an event on the continuous time torus representing that the state of  $\mathbf{X}$  consistent with the arrival times satisfies the understanding of  $E_2^{\text{crude}}$  above. Since

$$\mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(E_2^{\text{crude}}) \geq \mathbb{E} [\mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T}}(E_2 | \mathbf{X})] = \mathbb{P}_{\lambda_0, \tilde{p}, \lambda_1}^{\mathbb{T}}(E_2),$$

from (2.4.5) we have for infinitely many  $n$  that

$$\mathbb{P}_{\lambda, p+\varepsilon/2, 1}^{\mathbb{T} \times \{-1, 0, 1, 2, \dots, \lfloor n/\delta \rfloor\}}(E_2^{\text{crude}}) \geq \mathbb{P}_{\lambda, p+\varepsilon/2}^{\mathbb{T}}(E_2) \geq p^*/2. \tag{2.4.7}$$

Set  $N = 400n^2(2 + \lfloor \frac{n}{\delta} \rfloor)$  (a finite function of  $n$ ). Given a probability vector  $\underline{p}' = (p'_0, p'_1, p'_2, p'_3)$ , we define the probability measure  $\mathbb{P}_{\underline{p}'}^N$  on the space  $\{0, 1, 2, 3\}^N$  as in Section 2.2. We can now think of  $E_2^{\text{crude}}$  as being an event  $E_2^{\text{disc}}$  in  $\{0, 1, 2, 3\}^N$ , by enumerating the  $(x, k)$  pairs as  $z_1, z_2, \dots, z_N$  and identifying the value of  $\mathbf{X}$  with an element of  $\{0, 1, 2, 3\}^N$ . Given  $\lambda_0, \tilde{p}$  and  $\lambda_1$ , the distribution of  $\mathbf{X}$  under this identification is given by  $\mathbb{P}_{\underline{p}}^N$  with the entries of  $\underline{p}$  given by (2.4.6).

Since  $E_2$  is preserved by translations of the torus that send even sites to even sites, so too is  $E_2^{\text{crude}}$  preserved by translations of the underlying torus that send

even sites to even sites, and so has symmetry of order  $200n^2$ , as does  $E_2^{\text{disc}}$  under the induced permutations of the product space.

We claim that  $E_2^{\text{disc}}$  is increasing in  $\mathbf{X}$ . We shall argue this based on the nature of a pair  $z = (x, k) \in \mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}$ . If  $k = -1$  then  $z$  corresponds to the diamond site  $x'$ , and an increase in  $X(z)$  either corresponds to leaving  $x'$  unchanged, or changing  $x'$  from being white to being black. If  $k \geq 0$  and  $x$  is an odd site, an increase in  $X(z)$  from 0 corresponds to removing any arrivals at  $x$  in the time period  $I_\delta(k)$  and otherwise leaving things unchanged. If  $k \geq 0$  and  $x$  is an even site, an increase in  $X(z)$  corresponds to either leaving things unchanged, or creating an arrival at  $x$  in the time period  $I_\delta(k)$ . Thus regardless of the nature of a site  $z$ ,  $E_2^{\text{disc}}$  is increasing in  $X(z)$ .

In order to apply Proposition 2.2.1, we compare two models, i.e. two probability vectors  $(p_0, p_1, p_2, p_3)$  and  $(q_0, q_1, q_2, q_3)$ , where  $p_i$  is the probability that  $X(z) = i$  in the first model, and  $q_i$  is the probability that  $X(z) = i$  in the second model. Our first model has parameters  $\lambda_0 = \lambda$ ,  $\lambda_1 = 1$ , and  $\tilde{p} = p + \varepsilon/2$ , while our second model has parameters  $\lambda_0 = \lambda(1+\varepsilon)^{1/2}$ ,  $\lambda_1 = (1+\varepsilon)^{-1/2}$  and  $\tilde{p} = p + \varepsilon$ . Then using (2.4.6) we have

$$\begin{aligned} p_3 &= 1 - e^{-\lambda\delta}, & q_3 &= 1 - e^{-(1+\varepsilon)^{1/2}\lambda\delta}; \\ p_2 &= e^{-\lambda\delta} + \varepsilon/2 + p - 1, & q_2 &= e^{-(1+\varepsilon)^{1/2}\lambda\delta} + \varepsilon + p - 1; \\ p_1 &= e^{-\delta} - \varepsilon/2 - p, & q_1 &= e^{-(1+\varepsilon)^{-1/2}\delta} - \varepsilon - p; \\ p_0 &= 1 - e^{-\delta}, & q_0 &= 1 - e^{-(1+\varepsilon)^{-1/2}\delta}. \end{aligned}$$

From the correspondence with the torus we have

$$\mathbb{P}_{p_0, p_1, p_2, p_3}^N(E_2^{\text{disc}}) = \mathbb{P}_{\lambda, p+\varepsilon/2, 1}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(E_2^{\text{crude}}); \quad (2.4.8)$$

$$\mathbb{P}_{q_0, q_1, q_2, q_3}^N(E_2^{\text{disc}}) = \mathbb{P}_{(1+\varepsilon)^{1/2}\lambda, p+\varepsilon, (1+\varepsilon)^{-1/2}}^{\mathbb{T} \times \{-1, 0, 1, \dots, \lfloor n/\delta \rfloor\}}(E_2^{\text{crude}}). \quad (2.4.9)$$

From (2.4.8) and (2.4.7) we have that

$$\mathbb{P}_{p_0, p_1, p_2, p_3}^N(E_2^{\text{disc}}) \geq p^*/2.$$

We shall now apply Proposition 2.2.1. Note that

$$\begin{aligned}
q_3 - p_3 &= e^{-\lambda\delta} - e^{-(1+\varepsilon)^{1/2}\lambda\delta} \sim \delta((1+\varepsilon)^{1/2} - 1)\lambda; \\
p_1 - q_1 &= \varepsilon/2 + e^{-\delta} - e^{-(1+\varepsilon)^{1/2}\delta} \rightarrow \varepsilon/2; \\
p_0 - q_0 &= e^{-(1+\varepsilon)^{-1/2}\delta} - e^{-\delta} \sim \delta(1 - (1+\varepsilon)^{-1/2}) \\
&= \delta \frac{(1+\varepsilon)^{1/2} - 1}{(1+\varepsilon)^{1/2}}.
\end{aligned}$$

Set  $\gamma = \min(p_0 - q_0, q_3 - p_3)$ . For sufficiently high  $n$ , we obtain that  $p_0 > q_0$ ,  $p_1 > q_1$  and  $q_3 > p_3$ . Hence  $\gamma > 0$  and  $(q_0, q_1, q_2, q_3)$  dominates  $(p_0 - \gamma, p_1, p_2, p_3 + \gamma)$ . Then by (2.4.7) and (2.4.8), we may apply Proposition 2.2.1. In the terminology of that result, we have  $q_{\max} = \min(p_2, p_1)$ . Let  $\varepsilon_1 \in (0, \varepsilon^{40})$  and let

$$\eta = \min\{p^*/4, \varepsilon_1\}.$$

Since  $p \log(4/p)$  takes maximum value  $\log(4) < 2$ , the right hand side of (2.2.1) is at most  $4000 \log(1/\eta)$ . Since  $\delta = (\log n)^{-1/2}$ , for  $n$  large enough we have  $\gamma \log(200n^2) > 4000 \log(1/\eta)$ . Thus by Proposition 2.2.1,  $\mathbb{P}_{q_0, q_1, q_2, q_3}^N(E_2^{\text{disc}}) > 1 - \varepsilon_1$ , and hence from (2.4.9) we have

$$\mathbb{P}_{(1+\varepsilon)^{1/2}\lambda, p+\varepsilon, (1+\varepsilon)^{-1/2}}^{\mathbb{T} \times \{-1, 0, 1, 2, \dots, \lfloor n/\delta \rfloor\}}(E_2^{\text{crude}}) > 1 - \varepsilon_1. \quad (2.4.10)$$

Now consider any state  $\mathbf{X}_0 \in E_2^{\text{crude}}$ , let  $x_0, x_1, \dots$  form an enumeration of the sites of  $\mathbb{T} \cap \mathbb{Z}^2$ , let  $\text{col}_{x'}$  denote the colour of the diamond site  $x'$ , and let  $Z_1 = (\text{col}_{x'_0}, t_{x_0}, \text{col}_{x'_1}, t_{x_1}, \dots)$  be a collection of arrival times at octagon sites and colours of diamond sites on the torus which induces state  $\mathbf{X}_0$  and such that  $E_2$  holds. By definition, such a  $Z_1$  exists. Let  $Z_2$  be any other collection of octagon site arrival times and diamond site colours with state consistent with  $\mathbf{X}_0$ . At each even site of the torus, the first arrival time under  $Z_2$  can be at most  $\delta$  later than the first arrival at that site in  $Z_1$ , and similarly the first arrival at an odd site in  $Z_2$  can be no more than  $\delta$  earlier than the first arrival in  $Z_1$ . Therefore any sites which are black when all the arrival times at even sites in  $Z_1$  are delayed by  $2\delta$  (as per the definition of  $E_2$ ) are also black in  $Z_2$  (with no delay).

Since the existence of a horizontal crossing is increasing in black sites, and since  $Z_1$  with a  $2\delta$  delay on the arrival time at even sites has a horizontal black



crossing of some  $18n$  by  $2n$  rectangle,  $Z_2$  must therefore have a horizontal black crossing of the same  $18n$  by  $2n$  rectangle. Letting  $E_3$  be the event that there is a horizontal black crossing of some  $18n$  by  $2n$  rectangle in  $\mathbb{T}$ , by (2.4.10) and event inclusion we therefore have

$$\mathbb{P}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, (1+\varepsilon)^{-1/2}}^{\mathbb{T}}(E_3) > 1 - \varepsilon_1. \quad (2.4.11)$$

We now cover  $\mathbb{T}$  with a set of  $12n$  by  $4n$  rectangles  $R_1, \dots, R_{40}$  such that whenever  $E_3$  holds there is a black path crossing some  $R_i$  horizontally. We can do this by using rectangles with lower left corner having  $x$ -coordinate a multiple of  $5n$  and  $y$ -coordinate a multiple of  $2n$ . Let  $H_i$  be the event that  $R_i$  has a black horizontal crossing, and note that  $H_i^c$  is white-increasing. Using the FKG inequality, which holds for RSA in the (even) torus by section 5 of [22] and hence in this enhanced RSA model on an even torus by the independence of the enhancement variables, we have

$$\begin{aligned} \mathbb{P}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, \lambda(1+\varepsilon)^{-1/2}}^{\mathbb{T}} \left( \bigcap_{i=1}^{40} H_i^c \right) &\geq \prod_{i=1}^{40} \mathbb{P}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, \lambda(1+\varepsilon)^{-1/2}}^{\mathbb{T}}(H_i^c) \\ &= (\mathbb{P}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, \lambda(1+\varepsilon)^{-1/2}}^{\mathbb{T}}(H_1^c))^{40} \\ &= (1 - h_3(2n, \lambda(1+\varepsilon), p+\varepsilon) + o(1))^{40} \end{aligned}$$

by time rescaling, since  $h_3(2n, \lambda(1+\varepsilon), p+\varepsilon)$  is the probability in enhanced RSA on  $\mathbb{Z}^2$  with parameters  $\lambda(1+\varepsilon), p+\varepsilon$  that a  $12n$  by  $4n$  rectangle has a horizontal black crossing, and an associated event on the torus has probability within  $o(1)$ .

If none of the  $H_i$  hold then  $E_3$  fails, so by (2.4.11) we have

$$1 - h_3(2n, \lambda(1+\varepsilon), p+\varepsilon) \leq \varepsilon_1^{1/40} + o(1)$$

and thus by our choice of  $\varepsilon_1$ , we have (2.4.4). □

**Proposition 2.4.2.** *Let  $\lambda > 0$ ,  $p \in (0, 1)$  and  $\varepsilon > 0$ . Suppose for some  $\rho > 0$  that  $\limsup_{n \rightarrow \infty} h_\rho(n, \lambda, p) > 0$ . Then*

$$\limsup_{n \rightarrow \infty} h_3(n, \lambda + \varepsilon, p) > 1 - \varepsilon \quad (2.4.12)$$

and if  $p + \varepsilon < 1$  then

$$\limsup_{n \rightarrow \infty} h_3(n, \lambda, p + \varepsilon) > 1 - \varepsilon \quad (2.4.13)$$

*Proof.* By the first inequality of (2.4.3) we can find some absolute constant  $K$  such that for all  $n$  we have

$$h_3(n, \lambda, p + \varepsilon) \geq h_3(n, \lambda + \varepsilon/2K, p + \varepsilon/2)$$

and by Lemma 2.4.2, this exceeds  $1 - \varepsilon$  for infinitely many  $n$ , which gives us (2.4.13).

We prove (2.4.12) similarly, now using the second inequality of (2.4.3).  $\square$

## 2.5 Proof of Theorem 2.1.1

Using Propositions 2.4.1 and 2.4.2, we are nearly ready to prove our theorem. We first assemble some further facts based adapting known methods to eRSA.

**Lemma 2.5.1.** *There exists constants  $\kappa > 0$ , and  $n_0 \in \mathbb{N}$ , such that the even phase percolates if there exists  $n \geq n_0$  with  $h_3(n, \lambda, p) > 1 - \kappa$ .*

*Proof.* This can be proved by a similar method to Theorem 1.1 of [2], namely comparison with 1-dependent bond percolation along with use of Lemma 2.3.5.  $\square$

**Lemma 2.5.2.** *For eRSA with parameters  $(\lambda, p)$ , the even phase percolates if and only if*

$$\lim_{n \rightarrow \infty} h_3(n, \lambda, p) = 1;$$

*the odd phase percolates if and only if*

$$\lim_{n \rightarrow \infty} h_{1/3}(n, \lambda, p) = 0.$$

*Proof.* By Lemma 2.5.1, it is immediate that  $\limsup_{n \rightarrow \infty} h_3(n, \lambda, p) = 1$  implies percolation of the even phase.

Now, suppose that  $\liminf_{n \rightarrow \infty} h_3(n, \lambda, p) < 1$ ; it thus follows by symmetry

that  $\limsup_{n \rightarrow \infty} h'_{1/3}(n, \lambda, p) > 0$ . By Lemma 2.3.4, we then have

$$\limsup_{n \rightarrow \infty} h'_3(n, \lambda, p) > 0.$$

We can thus find a constant  $c > 0$  and a sequence  $(n_i)_{i \in \mathbb{N}}$  with  $n_{i+1} > 4n_i$  such that  $h'_3(n_i, \lambda, p) > c^{1/4}$ . Now, we consider the collection of rectangles  $(R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4})_{i \in \mathbb{N}}$  where we have

$$\begin{aligned} R_{i,1} &= [-3n_i, 3n_i] \times [-3n_i, -n_i] \\ R_{i,2} &= [n_i, 3n_i] \times [-3n_i, 3n_i] \\ R_{i,3} &= [-3n_i, 3n_i] \times [n_i, 3n_i] \\ R_{i,4} &= [-3n_i, n_i] \times [-3n_i, -3n_i]. \end{aligned}$$

We consider the collection of events  $(E_i)_{i \in \mathbb{N}}$  where  $E_i$  is the event that all of the rectangles  $R_{i,j}$  contain a long way crossing in the odd phase and that the events  $E_{\text{dense}}(R_{i,j}, n_i/8)$  hold. By Lemma 2.3.5, the probability  $\mathbb{P}_{\lambda,p}(E_{\text{dense}}(R_{i,j}, n_i/8)) \rightarrow 1$  as  $i \rightarrow \infty$ , and hence for  $i$  sufficiently high we have  $\mathbb{P}_{\lambda,p}(E_i) > c/2 > 0$ . Since the events  $(E_i)_{i \in \mathbb{N}}$  are independent, it follows that almost surely at least one of them occurs, and hence the cluster containing the origin in the even phase is almost surely finite.

Our proof is completed by symmetry and by noting that

$$\begin{aligned} \liminf_{n \rightarrow \infty} h_{1/3}(n, \lambda, p) &= 1 - \limsup_{n \rightarrow \infty} h'_3(n, \lambda, p), \\ \limsup_{n \rightarrow \infty} h_{1/3}(n, \lambda, p) &= 1 - \liminf_{n \rightarrow \infty} h'_3(n, \lambda, p). \end{aligned}$$

□

We shall refer to the following lemma as a *duality* relation.

**Lemma 2.5.3.** *Let  $\lambda > 0, p \in [0, 1]$ . Then the even phase of eRSA with parameters  $(\lambda, p)$  percolates, if and only if the odd phase of eRSA with parameters  $(1/\lambda, 1 - p)$  percolates.*

*Proof.* Consider first the eRSA process with parameters  $(\lambda, p)$ . Now re-scale time by multiplying all arrival times by a factor of  $\lambda$ ; the rescaled arrival times are exponential with rate 1 at even sites and rate  $1/\lambda$  at odd sites. If we then also

interchange the colours, then the new set of black sites is a realization of eRSA with parameters  $(1/\lambda, 1 - p)$ .  $\square$

Now we have all the tools needed to complete the proof of Theorem 2.1.1, using the strategy outlined in Section 2.1.

*Proof of Theorem 2.1.1.* Let  $n_0 \in \mathbb{N}$  and  $\kappa > 0$  be as in Lemma 2.5.1. Let  $S$  denote the set of  $(\lambda, p)$  such that  $h_3(n, \lambda, p) > 1 - \kappa$  for some  $n \geq n_0$ . Since  $h_3(n, \lambda, p)$  is continuous in  $\lambda$  and  $p$  for any fixed  $n$ , the set  $S$  is open in  $(0, \infty) \times [0, 1]$ , and the even phase percolates for any  $(\lambda, p) \in S$ . Also, if  $(\lambda, p) \notin S$  then  $\limsup(h_3(s, \lambda, p)) \leq 1 - \kappa < 1$ , and thus there is no percolation by Lemma 2.5.2. Hence,  $S$  is the set of  $(\lambda, p)$  for which the even phase percolates.

Similarly, with  $S'$  denoting the set of values  $(\lambda, p)$  for which the odd phase percolates, the set  $S'$  is also open in  $(0, \infty) \times [0, 1]$ . Also  $S \cap S' = \emptyset$  by Lemma 2.5.2. Since  $\lambda_c^+(p) = \inf\{\lambda : (\lambda, p) \in S\}$ , and  $\lambda_c^-(p) = \sup\{\lambda : (\lambda, p) \in S'\}$ , this gives us the inequality  $\lambda_c^-(p) \leq \lambda_c^+(p)$ .

For  $(\lambda, p), (\lambda', p') \in (0, \infty) \times [0, 1]$ , let us write  $(\lambda', p') \succ (\lambda, p)$  to mean that  $\lambda \leq \lambda'$  and  $p \leq p'$  with at least one of these inequalities being strict.

Suppose  $(\lambda, p) \notin S'$  and  $0 < p < 1$ . Then  $\limsup_{n \rightarrow \infty} h_{1/3}(n, \lambda, p) > 0$  by Lemma 2.5.2. Hence by Proposition 2.4.2, for any  $(\lambda', p') \succ (\lambda, p)$  we have  $\limsup_{n \rightarrow \infty} h_3(n, \lambda', p') = 1$ . Therefore  $(\lambda', p') \in S$  by Lemma 2.5.1. In other words, for  $0 < p < 1$  we have

$$(\lambda, p) \notin S' \implies (\lambda', p') \in S \quad \forall (\lambda', p') \succ (\lambda, p). \quad (2.5.1)$$

Hence for  $0 < p < 1$  we have  $\lambda_c^-(p) = \lambda_c^+(p)$ . Thus we have part (i) of our theorem, and part (ii) follows from the fact that the sets  $S$  and  $S'$  are open.

Since by symmetry  $h_1(n, 1, 1/2) = 1/2$  for all  $n$ , we have that  $\limsup h_3(n, 1, 1/2) < 1$ , so by Lemma 2.5.2 we have  $(1, 1/2) \notin S$ . Hence by duality, also  $(1, 1/2) \notin S'$ , and part (iii) follows.

For part (iv), the strict monotonicity of  $\lambda_c(\cdot)$  follows from (2.5.1) and the fact that  $S$  is open. We next prove the Lipschitz continuity of  $\lambda_c(\cdot)$ .

By Theorem 2.1 of [21],  $\lambda_c^+(0) < 10$ , and hence by duality,  $\lambda_c^-(1) > 0.1$ . By Proposition 2.4.1, we can find a strictly positive constant  $c_7$  such that for  $(\lambda, p, n) \in [0.1, 10] \times [0, 1] \times \mathbb{N}$ , we have the second inequality of (2.4.3).

Let  $p \in [\varepsilon, 1]$ . Then for any  $\lambda > \lambda_c^+(p)$ , we have  $\lambda \in S$  so we can find  $n \geq n_0$  such that  $h_3(n, \lambda, p) \geq 1 - \kappa$ . Then by the second inequality of (2.4.3), for such  $n$  and for  $0 < \delta < \varepsilon$  we have

$$h_3(n, \lambda + c_7\delta, p - \delta) \geq h_3(n, \lambda, p) \geq 1 - \kappa$$

so that  $(\lambda + c_7\delta, p - \delta) \in S$  and hence  $\lambda_c^+(p - \delta) \leq \lambda_c^+(p) + c_7\delta$ . This gives the Lipschitz continuity of  $\lambda_c^+(\cdot)$  on  $[\varepsilon, 1]$ .

Now suppose  $0 \leq p \leq 1 - \varepsilon$ . By duality (Lemma 2.5.3) we have  $\lambda_c^-(p) = 1/\lambda_c^+(1 - p)$ . Thus for  $0 < \delta < \min(\varepsilon, (100c_7)^{-1})$ , using that  $\lambda_c^+(1 - p) < 10$  we also have

$$\lambda_c^+(1 - p - \delta) \leq \lambda_c^+(1 - p) + c_7\delta \leq \frac{1}{\lambda_c^-(p) - c'\delta},$$

for  $c' = 100c_7$ . Hence by duality again,  $\lambda_c^-(p + \delta) \geq \lambda_c^-(p) - c'\delta$ . This shows the Lipschitz continuity of  $\lambda_c^-(\cdot)$  on  $[0, 1 - \varepsilon]$ .

For  $\lambda_c^+(1) < \lambda < \lambda_c^-(0)$ , set  $p_c^+(\lambda) := \inf\{p : (\lambda, p) \in S\}$ . By (2.5.1) and the fact that  $S$  is open, the function  $p_c^+(\cdot)$  is strictly decreasing. By a similar argument to the above (now using the first inequality of (2.4.3)), we may show the Lipschitz continuity of  $p_c^+(\lambda)$  as a function of  $\lambda$  for  $\lambda_c^+(1) + \varepsilon \leq \lambda \leq \lambda_c^-(0) - \varepsilon$ . Thus the restriction of the function  $\lambda_c^+(\cdot)$  to the domain  $[1 - \varepsilon, \varepsilon]$  has a Lipschitz inverse, namely  $p_c^+(\cdot)$ .  $\square$

## Chapter 3

# The Phase Transition in General Models

### 3.1 General Rotation Invariant Models

Presented in Duminil-Copin et al. (2014) [9] was a simple and powerful method for demonstrating a model does not permit percolation at any critical values. It does so by establishing a finite box criterion equivalent to percolation, which allows for continuity arguments to demonstrate that around any supercritical collection of parameters there is an open set of likewise supercritical parameters.

We deal here with a broad category of percolation models. We have a point process and a deterministic method of generating a collection of subsets of the plane given a realization of the point process. We say that a realization of this model percolates if among the generated collection of subsets of the plane there is at least one unbounded component. When we consider a subset of the model, we refer to the collection of randomly generated subsets of the plane, and when we consider the model restricted to some region we refer to the collection of subsets of the plane that would be generated by the point process restricted only to that region. We denote the point process by  $Z$ , and the collection of random subsets of the plane generated by  $Z$  by  $P_Z$ ; we generally omit  $Z$  when referring to the entire point process, and when  $P$  contains only a single subset we identify it with that subset. Such a model generally has one or more parameters linked to the point process; for instance the Gilbert disc model has a rate parameter  $\lambda$ .

Unlike the comparably broad method developed by Bollobás and Riordan in [2] (used above in Chapter 2), the method used does not require that geometrically crossing paths are connected almost surely, instead requiring a weaker statement, that the model behaves approximately as if crossing geometric paths are connected in one specific situation. It also permits a weaker asymptotic independence assumption; rather than requiring an event with high probability on a bounded region that implies independence of some contained region from the model outside the initial bounded region, we simply require that a specific event that occurs with high probability in the model also occur with high probability in the model restricted to only points of the point process within some surrounding region. The method does have one requirement not present in the original; roughly speaking a region of the model needs to be asymptotically increasing in the point process outside some larger containing region.

We seek to demonstrate a method by which many such models can be shown to

have a finite box criterion equivalent to percolation. As a consequence, any such model where finite box events are continuous in the parameters must necessarily have an open (potentially empty) set of parameters where percolation occurs and hence nonpercolation at any critical parameter set.

For such a model, for connected, measurable subsets of the plane  $R \subset R'$  with connected, measurable subsets  $S_1 \subset R$  and  $S_2 \subset R$  we denote by  $S_1 \xleftrightarrow[R']{R} S_2$  the event that  $P_{Z \cap R'} \cap R$  contains a component containing points in both  $S_1$  and  $S_2$ . We use  $\xleftrightarrow[R']{R!}$  to denote uniqueness of the component. We denote by  $S_1 \xleftrightarrow[R']{R} \partial R$  the event that  $P_{Z \cap R'} \cap R$  contains a component containing a point of  $S_1$  and a point of the boundary of  $R$  (under the standard Euclidean topology). For all of these, if  $R'$  is omitted it is assumed to be the entire space. We define the box  $B_n := [-n, n]^2$ .

Presented first are the broad results that provide the framework for this method, followed by applying this framework to a varied collection of models.

**Definition 3.1.1.** *For an integer  $k$ , we call a planar percolation model  $k$ -good if it satisfies the following:*

1. *The model is invariant under reflections of  $P$  in the  $x$  and  $y$  axes*
2. *The model is invariant under translations of  $P$  by an integer distance in the  $x$  and  $y$  axes*
3. *The model is invariant under rotations of  $P$  of  $\pi/2$  about the origin*
4. *In the model, for  $A \subset B$  bounded subsets of  $\mathbb{R}^2$ , events of the form  $R \xleftrightarrow[B]{A} R'$  are positively correlated*
5. *Whenever the model percolates there is an increasing sequence  $u_n$  such that all of the following hold*

$$(5a) \lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{!B_n!} \partial B_n \right) = 1,$$

$$(5b) \limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \{(0, n)\} \right) < 1,$$

$$(5c) \limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \{(n, n)\} \right) < 1,$$



(5d) *There is some event  $E_{\text{dense}} = E_{\text{dense}}(B_{kn})$  depending only on the point process within  $B_{kn}$  such that for any  $A \subset \mathbb{R}^2$ ,  $A \supset B_{kn}$  we have both of*

$$\lim_{n \rightarrow \infty} \mathbb{P}(E_{\text{dense}}) = 1,$$

$$\mathbb{P} \left( \bigcap_{R_{B_{kn}} \in P_{Z \cap B_{kn}}} \exists R_A \in P_{Z \cap A} : R_{B_{kn}} \cap B_n \subset R_A \cap B_n | E_{\text{dense}} \right) = 1.$$

In order to obtain our end result that for models meeting all our criteria we can find a finite box event equivalent to percolation we require one additional condition. The precise condition is given after Lemma 3.1.2. Broadly speaking our goal is to demonstrate that for any  $k$ -good model with the additional condition, we can build a collection of coupled discrete percolation model with at most finite range dependence, such that should any of the coupled models percolate then our original will also percolate, and that should our original percolate then at least one of the collection of discrete models will percolate (indeed that infinitely many will do so).

We shall be frequently using the so called square root trick, (seen for example in [4] p41).

**Lemma 3.1.1** (The Square Root Trick). *For positively correlated events  $A_1, \dots, A_n$ , we have*

$$\max_i \mathbb{P}(A_i) \geq 1 - (1 - \mathbb{P}(\cup_i A_i))^{\frac{1}{n}},$$

*and particularly in the case that  $\mathbb{P}(\cup_i A_i) = 1$  see that  $\max_i \mathbb{P}(A_i) = 1$ .*

The proof follows from simple manipulation of  $\mathbb{P}(\cup_i A_i) = 1 - \mathbb{P}(\cap_i A_i^c)$ , since the complement of a collection of positively correlated events is likewise positively correlated.

The proofs used in [9] were naturally rather broad, and require at most small alterations to adapt to our more general category of models. The goal with the collection of Lemmas is to be able to improve from condition (5a) to having with high probability a path between boxes  $B_{u_n}$ , which will then form the basis of our connection rule. In order to achieve this, we are going to apply the square root trick to the boundary of  $B_n$ , enabling us to find fractional sections of the boundary where nevertheless we reach the section with high probability; we want

to end with an interval reaching a corner of  $B_n$  and a second interval half the length of the first for each corner and each edge adjacent to it. By our rotation and reflection invariance, it is enough to find such for the right hand edge and upper corner, so we shall be working with that corner alone.

**Lemma 3.1.2.** *For any percolating  $k$ -good model, there exist sequences  $(a_n)_{n \in \mathbb{N}}$ , and  $(y_n)_{n \in \mathbb{N}}$  where  $0 < a_n < n$ , and  $0 < y_n < n - a_n/4$  such that the following hold for all  $\varepsilon > 0$ :*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} [a_n - \varepsilon, n] \times \{n\} \right) = 1,$$

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} [y_n - a_n/4 - \varepsilon, y_n + a_n/4 + \varepsilon] \times \{n\} \right) = 1.$$

*Proof.* We consider the boundary of the box  $B_n$ , and split it into 8 line segments, at each of the corners and midpoints. We then label these as  $A_1 := \{n\} \times [0, n]$  to  $A_8$  clockwise around the boundary of  $B_n$ . Since a combination of reflections through the horizontal and vertical axes and rotations by  $\pi/2$  around the origin can transform any  $A_i$  into any  $A_j$ , by invariance under reflection and rotation (conditions (1) and (3)) we see that for all  $i$  and  $j$  we have

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow{B_{kn}} A_i \right) = \mathbb{P} \left( B_{u_n} \xleftrightarrow{B_{kn}} A_j \right);$$

these events are all positively correlated by condition (4) and we can thus use the square root trick to produce

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow{B_{kn}} A_i \right) \geq 1 - (1 - \mathbb{P} \left( B_{u_n} \xleftrightarrow{B_{kn}} \partial B_n \right))^{1/8},$$

and by condition (5a) we see that  $\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow{B_{kn}} A_i \right) = 1$ .

We define

$$f_n(\alpha) := \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} [\alpha, n] \times \{n\} \right)$$

$$f'_n(\alpha) := \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} [0, \alpha] \times \{n\}, \right)$$

and by another application of the square root trick we see that

$$\max(f_n(\alpha), f'_n(\alpha)) \geq 1 - (1 - \mathbb{P}\left(B_{u_n} \xleftrightarrow[B_{kn}]{B_n} A_1\right))^{1/2}$$

and hence for any  $\alpha$  have that  $\lim_{n \rightarrow \infty} \max(f_n(\alpha), f'_n(\alpha)) = 1$ . By condition (5b) it follows that  $\lim_{n \rightarrow \infty} f'_n(0) < 1$ , and hence have that  $\lim_{n \rightarrow \infty} f_n(0) = 1$  and then for sufficiently high  $n$  see that  $f_n(0) > f'_n(0)$ . We can similarly argue that for sufficiently high  $n$  we have  $f_n(n) < f'_n(n)$ . Combining these, should  $n$  be sufficiently large that both inequalities hold then by the monotonicity of  $f_n$  and  $f'_n$  we can find some  $0 < \alpha_n < n$  such that for  $\beta < \alpha_n$  we have  $f_n(\beta) \geq f'_n(\beta)$ , and for  $\beta > \alpha_n$  we have  $f_n(\beta) \leq f'_n(\beta)$ .

We then let  $\alpha_n := \beta_n - \varepsilon_n$  to see that

$$\lim_{n \rightarrow \infty} f_n(\alpha_n) = \lim_{n \rightarrow \infty} \max(f_n(\alpha_n), f'_n(\alpha_n)) = 1.$$

Finally, we split the interval  $[0, \alpha_n + \varepsilon]$  in half and apply the square root trick once more, to achieve that at least one of  $y_n = \alpha_n/4$  and  $y_n = 3\alpha_n/4$  will satisfy our requirements.

□

Having found  $(a_n)$  and  $(y_n)$  we define

$$\begin{aligned} Z_n(\varepsilon) &:= \{3n\} \times [y_{3n} - a_{3n}/4 - \varepsilon, y_{3n} + a_{3n}/4 + \varepsilon], \\ Y_n^-(\varepsilon) &:= \{3n\} \times [y_{3n} - n, y_{3n} - a_n + \varepsilon], \\ Y_n^+(\varepsilon) &:= \{3n\} \times [y_{3n} + a_n - \varepsilon, y_{3n} + n], \\ B'_{u_n} &:= B_{u_n} + (2n, y_{3n}), \\ B'_n &:= B_n + (2n, y_{3n}). \end{aligned}$$

We also define

$$\begin{aligned} p_1 := \limsup_{n \rightarrow \infty} \sup_{\varepsilon > 0} \mathbb{P} &\left( B_{u_{3n}} \xleftrightarrow[B_{4kn}]{B_{4n}} B'_{u_n} \cap \mathbb{1}_{a_{3n} < 4a_n} \right. \\ &\left. | B_{u_{3n}} \xleftrightarrow[B_{4kn}]{B_{3n}} Z_n(\varepsilon), B'_{u_n} \xleftrightarrow[B_{4kn}]{B'_n} Y_n^-(\varepsilon), B'_{u_n} \xleftrightarrow[B_{4kn}]{B'_n} Y_n^+(\varepsilon) \right). \end{aligned} \quad (3.1.1)$$

This probability determines how well we're able to stitch together boxes, and

hence will give a lower bound on the probability of an edge being included in our eventual coupled models. Note that if all of the crossings being conditioned upon exist, then we can use the paths  $B'_{u_n} \xleftrightarrow[B_{4kn}]{B_n + (2n, y_{3n})} Y_n^-$  and  $B'_{u_n} \xleftrightarrow[B_{4kn}]{B_n + (2n, y_{3n})} Y_n^+$ , the boundary  $\partial B'_{u_n}$ , and the section of the edge  $\{3n\} \times [y_{3n} - n, y_{3n} + n]$  between the intersections with the paths from  $B'_{u_n}$  to divide  $B_{3n} \cup B'_{u_n}$  into 3 sections; then a path  $B_{u_{3n}} \xleftrightarrow[B_{4kn}]{B_{3n}} Z_n$  must cross between sections at some point, and thus must either pass through  $B'_{u_n}$  or cross one of the paths we chose. As such, if geometrically crossing paths merge almost surely then we can see that  $p_1 = 1$ .

**Corollary 3.1.1.** *For a percolating  $k$ -good model there are infinitely many  $n$  such that  $a_{3n} < 4a_n$ , and for any  $\varepsilon > 0$*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{4kn}]{B_{3n}} Z_n(\varepsilon), B'_{u_n} \xleftrightarrow[B_{4kn}]{B_n} Y_n^-(\varepsilon), B'_{u_n} \xleftrightarrow[B_{4kn}]{B_n} Y_n^+(\varepsilon) \cup \mathbb{1}_{a_{3n} \geq 4a_n} \right) = 1. \quad (3.1.2)$$

*Proof.* We can find infinitely many  $n$  such that  $a_{3n} < 4a_n$ . To see this, let  $m$  be any integer, and consider the collection  $x_n := a_{3^n m}$ . Should  $x_{n+1} \geq 4x_n$  for all  $n$ , we see that  $x_{n+1} \geq 4^n x_0$ . However since  $0 < a_n < n$ , we have  $x_{n+1} < 3^n m$  and  $x_0 > 0$ . Then letting  $n$  be sufficiently high we reach a contradiction. The result then follows Lemma 3.1.2 and condition (5d).  $\square$

We remain in the very general collection of proofs used in [9], and now have enough to calculate the probability of a local path connecting a pair of nearby boxes. Note that this can be easily adapted to giving an edge crossing result (specifically, if the event mentioned occurs then there must be a horizontal crossing of the box  $[u_{3n}, 4n - u_{3n}] \times [-3n, 3n]$ ), allowing for more conventional methods based on box crossings.

The idea behind the proof here is to set up a collection of crossings as in Corollary 3.1.1, so that  $B_{u_{3n}}$  connects to  $B'_{u_n}$  with probability close to  $p_1$ ; reflect the system around the line  $\{2n\} \times \mathbb{R}$  to see that  $B_{u_{3n}} + (4n, 0)$  connects to  $B'_{u_n}$  with the same probability, and then apply condition (5a) to  $B'_{u_n}$  to argue that with high probability there is only one large component leaving  $B'_{u_n}$ , hence the paths to  $B_{u_{3n}}$  and  $B_{u_{3n}} + (4n, 0)$  must meet somewhere.

**Corollary 3.1.2.** *For a percolating  $k$ -good model we have*

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{6n} + (2n, 0)} B_{u_{3n}} + (4n, 0) \right) \geq p_1^2. \quad (3.1.3)$$

*Proof.* Consider the event that in the model restricted to  $B_{6kn} + (2n, 0)$  there is simultaneously a path from  $B_{u_{3n}}$  to  $B'_{u_n}$ , a path from  $B_{u_{3n}} + (4n, 0)$  to  $B'_{u_n}$ , and that there is a unique cluster in  $B_n + (2n, y_{3n})$  that contains a point of  $B'_{u_n}$  and a point of the boundary  $\partial B_n + (2n, y_{3n})$ . Clearly, should this happen then there must be a path connecting  $B_{u_{3n}}$  to  $B_{u_{3n}} + (4n, 0)$ . Then we have

$$\begin{aligned} & \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{6n} + (2n, 0)} B_{u_{3n}} + (4n, 0) \right) \\ & \geq \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \cap B_{u_{3n}} + (4n, 0) \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \cap \right. \\ & \quad \left. \cap B'_{u_n} \xleftrightarrow[B_{6kn} + (2n, 0)]{!B'_n!} \partial B'_n \right) \\ & \geq \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \cap B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \right) + \\ & \quad + \mathbb{P} \left( B'_{u_n} \xleftrightarrow[B_{6kn} + (2n, 0)]{!B'_n!} \partial B'_n \right) - 1 \\ & \geq \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \right)^2 + \\ & \quad + \mathbb{P} \left( B'_{u_n} \xleftrightarrow[B_{6kn} + (2n, 0)]{!B'_n!} \partial B'_n \right) - 1. \end{aligned}$$

The last inequality follows by condition (4).

By Bayes' rule, Corollary 3.1.1, and the definition of  $p_1$  (3.1.1) we see that for any  $\varepsilon' > 0$  we have

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_{3n}} \xleftrightarrow[B_{6kn} + (2n, 0)]{B_{4n}} B'_{u_n} \right) \geq p_1 - \varepsilon'.$$

Finally we use condition (5a) to achieve our result. □

We now have the framework required to state and prove our main theorem for this section. As mentioned at the start, we shall construct a collection of linked discrete percolation models; specifically we shall construct one model for each  $n$

such that  $a_{3n} < 4a_n$ , with points of the integer lattice associated to translations of  $B_{u_{3n}}$  and edges associated to local connections between these boxes with the additional rule that if there is not a unique path from  $B_{u_{3n}}$  to  $\partial B_{3n}$  then we forbid any edge from the associated point, and similarly forbidding edges if the (local) event that causes (local) paths to be preserved under adding points at distance fails to occur. In this way our edges only depend on local conditions, the associated paths to edges must meet if the edges do, and the various paths produced are all preserved when the remainder of the model is added.

Since everything is local, our coupled models all have finite range dependence, so we can find a nontrivial critical value for them. Then all that remains is showing that when our model percolates, one of the coupled models must likewise percolate.

**Theorem 3.1.1.** *For a  $k$ -good planar percolation model there is a constant  $p_k < 1$  depending only on  $k$  such that whenever  $p_1 > p_k$ , we can find  $q_k$  and produce a collection of events  $E(B_n)$  where  $E(B_n)$  depends only on the state of the point process within  $B_n$  such that should we have some  $n$  satisfying*

$$\mathbb{P}(E(B_n)) > q_k$$

*then the model percolates; and whenever the model percolates we can find  $n$  satisfying  $\mathbb{P}(E(B_n)) > q_k$ .*

*Additionally we see that if  $\mathbb{P}(E(B_n))$  is continuous in the parameters of the model for all  $n$ , then the set of parameters for which the model percolates is open.*

*Proof of Theorem 3.1.1.* In order to achieve this, we produce a coupled discrete bond percolation model with finite range dependence.

For any  $n$ , we can produce a bond percolation model on  $4n\mathbb{Z}^2$  by including

the edge  $\{z, z'\}$  whenever the following all hold:

$$B_{u_{3n}} + z \xleftrightarrow[B_{6kn} + \frac{z+z'}{2}]{B_{6n} + \frac{z+z'}{2}} B_{u_{3n}} + z', \quad (3.1.4)$$

$$B_{u_{3n}} + z \xleftrightarrow[B_{3kn}]{!B_{3n}} \partial B_{3n} + z \text{ and } B_{u_{3n}} + z' \xleftrightarrow[B_{3kn}]{!B_{3n}} \partial B_{3n} + z', \quad (3.1.5)$$

$$E_{\text{dense}}(B_{3kn} + z'), \quad (3.1.6)$$

$$E_{\text{dense}}(B_{3kn} + z), \quad (3.1.7)$$

$$E_{\text{dense}}(B_{6kn} + \frac{z+z'}{2}). \quad (3.1.8)$$

The edges  $\{x, y\}$  and  $\{x', y'\}$  are independent whenever there is no intersection between  $B_{6kn} + \frac{x+y}{2}$  and  $B_{6kn} + \frac{x'+y'}{2}$ ; this in turn happens whenever the objects are at graph distance at least  $6k$ . Our constructed discrete model is thus a  $6k$ -dependent percolation model; by a classical argument (see [16]) there is a constant  $q_k < 1$  such that for any  $6k$ -dependent discrete bond percolation model, whenever the marginal probability of an edge being included is at least  $q_k$  there is percolation. Then letting  $E(B_{6kn} + (2n, 0))$  be the event that the edge  $\{(0, 0), (0, 1)\}$  is included, we see that we have a collection of events each depending only on the model within some finite box and such that if  $\mathbb{P}(E(B_{6kn} + (2n, 0))) > q_k$  we have that the coupled model percolates.

Now we let  $p_k = \sqrt{q_k}$ , choose parameters such that our model percolates and has  $p_1 > p_k$ , and let  $\varepsilon := p_1^2 - q_k > 0$ . By corollary 3.1.2, we can find arbitrarily large  $n$  such that the probability of (3.1.4) exceeds  $q_k + 2\varepsilon/3$ ; by condition (5a) we can then find such an  $n$  which also satisfies

$$\mathbb{P}\left(B_{u_{3n}} + z \xleftrightarrow[B_{3kn}]{!B_{3n}} z + \partial B_{3n}\right) \geq 1 - \varepsilon/3.$$

Finally, by condition (5d) we can find an  $n$  such that

$$\mathbb{P}(E_{\text{dense}}) > 1 - \varepsilon/3.$$

With such an  $n$  chosen, we see that  $E(B_{6kn} + (2n, 0)) > q_k$ .

We now note that the probability that a specific edge  $\{z, z'\}$  is included depends only on the state of the model within the finite box  $B_{6kn} + \frac{z+z'}{2}$ . By the continuity of the probability of finite box events, it follows that the set of param-

eters for which this model percolates is open, and hence there is a neighbourhood of our initial parameters for which our discrete model percolates.

It remains to prove that the discrete model percolating implies the coupled model percolates. We claim that if there is a path between  $z$  and  $w$  in the discrete model, then we have the event  $B_{u_{3n}} + z \leftrightarrow B_{u_{3n}} + w$ , and we prove this by induction on the path length. Should  $w$  be at path length 1 from  $z$ , we can without loss of generality assume that  $w = z + (4n, 0)$ . Then since the edge  $\{z, w\}$  exists we have by definition and by condition (5d) that  $B_{u_{3n}} \leftrightarrow B_{u_{3n}} + (4n, 0)$ . Now we assume that the result holds for path length at most  $M$ , and pick  $v$  on the shortest path between  $z$  and  $w$ . By our assumption we have  $B_{u_{3n}} + z \leftrightarrow B_{u_{3n}} + v$  and  $B_{u_{3n}} + v \leftrightarrow B_{u_{3n}} + w$ . By condition (5d) and since we have  $E_{\text{dense}}(B_{3kn} + v)$ , with probability 1 we have that the state of the model within  $B_{3n} + v$  is the unchanged by discarding all points of the point process outside  $B_{3kn} + v$ ; and then since  $B_{u_{3n}} + v \xleftrightarrow[B_{3kn} + v]{B_{3n} + v} \partial B_{3n}$  we see that any paths in the model as a whole that contain points outside  $B_{3n} + v$  and points within  $B_{u_{3n}} + v$  must be connected. Finally we note that  $u_{3n} + 3n < 4n$  and so there can be no intersection between  $B_{u_{3n}} + v'$  and  $B_{3n} + v$  for  $v' \neq v$ . With all these, it follows that there is a path such that  $B_{u_{3n}} + z \leftrightarrow B_{u_{3n}} + w$ . It thus follows that whenever there is an unbounded component in the discrete model, there is an unbounded component in our original percolation model.

Finally we define  $f_n = \mathbb{P}(E(B_n))$  as a function in the parameters, the preimage  $f_n^{-1}(p, 1]$  will contain parameters that percolate, and since for any set of parameters where there is percolation we can find  $n$  such that  $\mathbb{P}(E(B_n)) > q_k$  we see that any parameters that percolate must be contained in  $\Lambda = \bigcup_n f_n^{-1}(q_k, 1]$ . Since  $(p, 1]$  is open in  $[0, 1]$  then by the continuity of  $f_n$  we see that  $\Lambda$  is an open set.  $\square$

To demonstrate the broad applicability of these results, we shall demonstrate that eRSA (as in Chapter 2.1) is 2-good, as is the Poisson Boolean model provided the distribution of the size of the balls produced has finite second moment.

To see eRSA is a model per our definition, we construct it as follows. We have a deterministic marked point process in which a point is placed at each point of  $\mathbb{Z}^2$  and at each point of  $\mathbb{Z}^2 + (0.5, 0.5)$ , where each point at an even integer site is marked with an independent exponential random variable of rate  $\lambda$ , each point at an odd integer site is marked with an independent exponential random variable



of rate 1, and each point not at an integer site is marked with a Bernoulli( $p$ ) random variable. We then consider the sites as octagons and diamonds as in Section 2.3, and declare  $P$  to be those octagons and diamonds which form the even phase.

We construct the Poisson Boolean model as follows. We have a single parameter  $\lambda$ , and a size distribution  $X$ . We then let  $Z$  be a homogeneous Poisson Point process on  $\mathbb{R}^2$  with density  $\lambda$ , and to each point  $z$  of  $Z$  we attach a radius value  $r_z$  which is distributed according to  $X$  independently of all other points. We then produce a random subset of the plane by attaching to each point  $z$  a ball of radius  $r_z$ .

The majority of the conditions for a model to be  $k$ -good are easily achieved; the only two requiring significant proof are that there is some event  $E_{\text{dense}}$  with the required properties and that we can find some  $u_n$  and  $k$  such that  $B_{u_n} \xleftrightarrow[B_{kn}]{{!B_n!}} \partial B_n$ . As a general result to aid with proving the second condition, it can be demonstrated that for any model in which there is almost surely precisely one infinite cluster then  $\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{{!B_n!}} \partial B_n \right) = 1$ . The idea of the proof is to choose  $u_n$  small enough relative to  $n$  that with high probability there is at most one component of the infinite cluster intersected with  $B_n$  which intersects  $B_{u_n}$ ; small enough relative to  $n$  that with low probability that there is a finite cluster that intersects  $B_{u_n}$  and intersects  $\partial B_n$ ; and large enough that with high probability the infinite cluster intersects  $B_{u_n}$ . With this result, condition (5a) can be proved by for example finding some  $k$  such that with high probability, the model outside  $B_{kn}$  is independent of the model inside  $B_n$ .

**Lemma 3.1.3.** *For any random subset of the plane such that there is almost surely either one infinite cluster or no infinite clusters, there is an increasing sequence  $(u_n)_{n \in \mathbb{N}}$  where  $u_n < n/3$  such that for any rectangle  $R \supset B_n$  the following holds when the model contains an infinite cluster*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow{{!R!}} \partial R \right) = 1.$$

*Proof.* Let  $m \in \mathbb{N}$  and let  $A_{n,m}$  be the event that there is at most one component of the infinite cluster intersected with  $B_n$  which has nonempty intersection with  $B_m$  and that  $n \geq m$ . Since  $R \supset B_n$ , it follows that whenever  $A_{n,m}$  holds there is also at most one component of the infinite cluster intersected with  $R$  which

has nonempty intersection with  $B_m$ . By monotonicity it can be seen that for any fixed  $m$  that  $\lim_{n \rightarrow \infty} \mathbb{P}(A_{n,m}) = 1$ . We then let  $v_{n,\varepsilon} = \sup\{m : \mathbb{P}(A_{n,m}) > 1 - \varepsilon\}$ , and note that  $\lim_{n \rightarrow \infty} v_{n,\varepsilon} = \infty$  for any fixed  $\varepsilon$  and that  $v_{n,\varepsilon}$  is increasing in  $n$ . Now let  $r_0 = 0$ , iteratively define  $r_k$  such that  $v_{r_k, 2^{-k}} > v_{r_{k-1}, 2^{1-k}} + 1$ , and define a sequence  $(v_n)_{n \in \mathbb{N}}$  by  $v_n = v_{k(n), 2^{-k(n)}}$  where  $k(n) = \max(k : r_k < n)$ . Note that  $k \rightarrow \infty$  and hence  $v_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Let  $E_n$  be the event that there is a finite cluster containing a point inside  $[-1, 1]^2$  which goes outside  $[-n/2, n/2]^2$ , and let  $p_n := \mathbb{P}(E_n)$ ; by monotonicity,  $p_n \rightarrow 0$ . Now we let  $u_n = \left\lfloor \min(n/3, p_n^{-1/4}, v_n) \right\rfloor$  and divide  $B_{u_n}$  into squares of side length 2. For each one of these squares, the probability that they are in a finite cluster which has points outside  $R$  is at most  $p_n$ , and then by the union bound since there are no more than  $p_n^{-1/2}/2$  squares the probability that there is at least one square containing points of a finite cluster going outside  $B_n$  is no more than  $p_n^{1/2}/2 \rightarrow 0$  by monotonicity.

Since  $p_n \rightarrow 0$  it follows that  $u_n \rightarrow \infty$  and hence that the probability that the infinite cluster considering the model as a whole has nonempty intersection with  $B_{u_n}$  tends to 1 whenever the model contains an infinite cluster. Then by the uniqueness of the infinite cluster, with high probability all clusters containing points in  $B_n$  that connect outside  $B_n$  and contain points of  $B_{u_n}$  are in the same cluster of the model as a whole. By our choice of  $v_n$  we also see that with high probability, the infinite cluster has at most one component in  $B_n$  which intersects  $B_{u_n}$ .

Then the result follows. □

For eRSA and the Poisson Boolean model, the majority of the conditions are immediate. The remaining steps are to prove our asymptotic invariance requirement for the Poisson Boolean model, and for eRSA to alter it to allow for integer translations.

**Theorem 3.1.2.** *Enhanced RSA is equivalent to a 2-good model with  $p_1 = 1$ .*

*Proof.* That our construction of eRSA model satisfies conditions (1) and (3) is immediate; that eRSA satisfies (4) is a consequence of such paths being decreasing in the arrival times at even sites and increasing in the arrival times at odd sites, and hence from the FKG inequality which holds for eRSA by section 5 of [22]. By

Lemma 2.3.5 there is an event  $E_{\text{dense}}(R_{n,\rho}, \lfloor n^{1/2} \rfloor)$  depending only on the state of the points of the model within  $R_{n+n^{1/2},\rho}$  such that should  $E_{\text{dense}}$  occur then  $R_{n,\rho}$  is independent of the model outside  $R_{n+n^{1/2},\rho}$ ; since  $E_{\text{dense}}(R_{n,\rho}, \lfloor n^{1/2} \rfloor)$  occurs with high probability and eventually  $n + n^{1/2} < kn$  for any  $k > 1$  we have condition (5d).

It can be shown by a Burton-Keane style argument (an example of which can be seen for instance in [4], p117 onwards) that eRSA almost surely has one or zero infinite components, and hence by Lemma 3.1.3 we have some  $(u_n)$  such that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n \right) = 1.$$

It then follows that

$$\begin{aligned} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n \right) &\geq \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n \cap E_{\text{dense}} \right) \\ &\geq \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n \right) + \mathbb{P}(E_{\text{dense}}) - 1, \end{aligned}$$

and since  $\lim_{n \rightarrow \infty} \mathbb{P}(E_{\text{dense}}) = 1$  we can conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n \right) = 1$$

and thus obtain condition (5a)

In order to have a model equivalent to our construction of eRSA satisfy condition (2), we simply rescale the model by a factor of  $1/2$ . It is then clear that a translation by an integer distance in this rescaled model maps even sites onto even sites, odd sites onto odd sites, and enhancement sites onto enhancement sites, and hence preserves the model. This rescaling does not alter any of the prior conditions, so they still hold in the shrunk model.

Finally, we note that the probability that in the shrunk model the site  $(n, n)$  is part of the even phase is independent of  $n$  and is equal to the probability in the shrunk model that  $(0, n)$  is part of the even phase; this is less than  $1 - \frac{1}{4\lambda+1}$  which can be seen by considering the event that of the (even) sites at  $(0, n)$ ,  $(1, n)$ ,  $(1/2, n - 1/2)$  and  $(1/2, n + 1/2)$  and the (odd) site at  $(1/2, n)$  the first arrival is at  $(1/2, n)$ ; should this happen then we must have that arrivals at  $(0, n)$  are blocked, and hence the site is in the odd phase. With this we have conditions

(5c) and (5b).

Thus eRSA satisfies all conditions other than condition (2). By rescaling eRSA by a factor of  $1/2$  in both directions, we achieve that integer translations send even sites to even sites and odd sites to odd sites, and hence this rescaled eRSA is a 2-good model. Lastly we note that by construction of eRSA, should two paths intersect geometrically they must be connected, and hence  $p_1 = 1$ .  $\square$

For the Poisson Boolean model, the only condition that presents any difficulty is condition (5a). For our formulation of the Poisson Boolean it suffices to show that the probability of a point outside  $B_{kn}$  having nonempty intersection with  $B_n$  tends to 0 as  $n \rightarrow \infty$ ; however we present a more robust proof that additionally covers the variant of the Poisson Boolean model where instead of letting  $P$  be the collection of disks, we have  $P$  be the collection of edges between center points of disks that have nonempty intersection. In this model, we would also need to account for the possibility that the only paths from  $B_{un}$  to a point outside  $B_n$  use a connection from a point of  $B_n$  to a point outside  $B_{kn}$  even if the intersection between the relevant disks happens outside  $B_n$ .

We first produce a bound on the probability that we have a point of  $B_n$  and of  $B_{2n}$  such that these points are not adjacent, but the point of  $B_n$  is adjacent to a point outside  $B_{6n}$ . The requirement that the points not be adjacent works to give a lower bound on the size of the disk generated outside  $B_{6n}$ , and will then be used to argue that the expected number of points making a large enough disk tends to 0.

**Lemma 3.1.4.** *For the Poisson-Boolean model with  $X$  such that  $\mathbb{E}[X^2] < \infty$ , define  $A'_{n,m}$  as being the event that there are points  $p_1, p_2$  and  $p_3$  such that  $p_1$  is within  $B_n$ , that  $p_2$  is within  $B_{2n} \setminus B_n$ , that  $p_3$  is outside  $B_m$ , that  $p_1$  and  $p_2$  are not adjacent, and that  $p_1$  is adjacent to  $p_3$ . Then  $\mathbb{P}(A'_{n,5n}) \rightarrow 0$ .*

*Proof.* In order that  $p_1$  and  $p_2$  are not adjacent, it must be the case that  $r_1 + r_2 < 3\sqrt{2}n$ . Thus we see that for any  $p_3$  at distance  $d + 5n$  from the edge of the box  $B_n$ , we must have  $r_3 > d + (5 - 3\sqrt{2})n$ . We can then consider the points  $p$  which satisfy  $d(p, B_n) > 5n$  and  $r > d(p, B_n) - 3\sqrt{2}n$  as being a Poisson point process, and calculate the expected number of points of this process (as an upper bound on the probability that there are any points) by taking the intensity measure of

the space.

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda \mathbb{P}(X > d((x, y), B_n) - 3\sqrt{2}n) \mathbb{1}_{(x, y) \notin B_{5n}} dx dy \leq \\
& \leq 4\lambda \int_{5n}^{\infty} \int_{5n}^y \mathbb{P}(X > y - 3\sqrt{2}n) dx dy \leq \\
& = 4\lambda \int_{5n}^{\infty} (y - 5n) \mathbb{P}(X > y - 3\sqrt{2}n) dy \\
& \leq 4\lambda \int_{5n}^{\infty} (y - 3\sqrt{2}n) \mathbb{P}(X > y - 3\sqrt{2}n) dy \\
& = 4\lambda \int_{(5-3\sqrt{2})n}^{\infty} x \mathbb{P}(X > x) dx.
\end{aligned}$$

Then since  $\int_0^{\infty} x \mathbb{P}(X > x) dx = \mathbb{E}[X^2] < \infty$  and  $5 > 3\sqrt{2}$  it follows that

$$\lim_{n \rightarrow \infty} \int_{(5-3\sqrt{2})n}^{\infty} x \mathbb{P}(X > x) dx = 0,$$

and our result follows.  $\square$

We now consider the model within a box  $B_{2n}$ . Since there is a high probability that there are points in the annulus  $B_{2n} \setminus B_n$ , we combine this with the previous lemma to see that with high probability, any point that is adjacent to a point outside  $B_{5n}$  is also adjacent to a point within  $B_{5n}$ , and thus we do not meaningfully impact the probability of  $B_{u_n} \xleftrightarrow{!B_n!} \partial B_n$  by discarding all points outside  $B_{5n}$ .

**Lemma 3.1.5.** *For a percolating Poisson-Boolean model with  $X$  such that  $\mathbb{E}[X^2] < \infty$ , there is an increasing sequence  $(u_n)_{n \in \mathbb{N}}$  with  $u_n \leq n/3$  such that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{5n}]{!B_n!} B_{5n} \setminus B_n \right) = 1.$$

*Proof.* The proof follows from Lemmas 3.1.3 and 3.1.4.

By Lemma 3.1.3 and by the uniqueness of the infinite cluster in the Poisson-Boolean model (see for example [18]) we can find a sequence  $(u_n)$  such that with high probability there is a unique cluster in the model restricted to  $B_n$  which contains a point adjacent to a point outside  $B_n$  and a point inside  $B_{u_n}$ .

We let  $E_1$  be the event that the square annulus  $B_{2n} \setminus B_n$  contains at least one point; since the area of  $B_{2n} \setminus B_n$  is  $12n^2$  it follows that  $\lim_{n \rightarrow \infty} \mathbb{P}(E_1) = 1$ .

Thus with probability tending to 1 we simultaneously have the events  $E_1$  and  $E_2 := B_{u_n} \xleftrightarrow{!B_n!} \partial B_n$ . Should  $E_2$  occur, we can produce a nonempty set  $Z$  of points outside  $B_n$  which are adjacent to a point of the unique cluster restricted to  $B_n$  that contains a point of  $B_{u_n}$  and a point adjacent to a point outside  $B_n$ .

We first consider the case that  $Z$  contains a point of  $B_{5n} \setminus B_n$ ; in this case the event  $B_{u_n} \xleftrightarrow[B_{5n}]{!B_n!} B_{5n} \setminus B_n$  occurs immediately.

In the case that  $Z$  does not contain any points of  $B_{5n} \setminus B_n$  and  $E_1$  occurs we can find adjacent points  $x \in B_n$  and  $z \notin B_{5n}$  such that the cluster restricted to  $B_{5n}$  containing  $x$  also contains points of  $B_{u_n}$ , and a point  $y \in B_{2n} \setminus B_n$  such that  $y$  is not adjacent to  $x$ . However, by Lemma 3.1.4 the probability that we have such  $x, y, z$  tends to 0 as  $n \rightarrow \infty$ .

Since  $\lim_{n \rightarrow \infty} \mathbb{P}(E_1) = 1$ , the final case has probability likewise tend to 0 as  $n \rightarrow \infty$ .

Thus with high probability we simultaneously have the events  $E_1$ ,  $E_2$ , and  $Z \cap B_{5n} \setminus B_n \neq \emptyset$ . The result follows.  $\square$

**Theorem 3.1.3.** *Any Poisson Boolean model such that  $\mathbb{E}[X^2] < \infty$  is a 5-good model and has  $p_1 = 1$ .*

*Proof.* Once more, conditions (1), (2), (3) and (4) are satisfied immediately. Letting  $k = 5$  we have condition (5a) from Lemma 3.1.5. For conditions (5c) and (5b), we consider the probability that an arbitrary point is contained within a ball. As shown in [17][7.3], this is less than 1 for finite  $\lambda > 0$  and  $\mathbb{E}[X^2] < \infty$ . Since the probability of the event considered in condition (5c) is at most the probability that  $(n, n)$  is contained in at least one ball, which is in turn bounded by the probability that  $(0, 0)$  is contained within a ball in the whole model, our probability is bounded away from 1 and condition (5c) is satisfied. An equivalent argument applies for condition (5b).

Finally, since  $P_{Z \cap B_{kn}} \subset P$  by construction of the model, all conditions are satisfied; and since geometrically crossing paths are connected we have  $p_1 = 1$  automatically.  $\square$

**Corollary 3.1.1.** *The Poisson Boolean model with balls of random radius has no percolation at criticality for any radius distribution.*

*Proof.* By Theorems 3.1.1 and 3.1.3 we see that provided  $\mathbb{E}[X^2] < \infty$ , the set of  $\lambda$  for which the Poisson Boolean model percolates is open. It then follows that for  $\lambda_c$  such that there is no percolation whenever  $\lambda < \lambda_c$  and there is percolation whenever  $\lambda > \lambda_c$  we cannot have percolation at  $\lambda_c$  provided  $\lambda_c < \infty$ . It is easy to see by comparison to a rescaled Gilbert disk model that  $\lambda_c < \infty$  whenever  $\mathbb{E}[X] > 0$ . Finally, for the case that  $\mathbb{E}[X] = 0$ , our random subset of the plane is the emptyset, hence there is no percolation at  $\lambda_c = \infty$ .

For the case that  $\mathbb{E}[X^2] = \infty$ , as seen in [17] (Proposition 7.3), if  $\lambda > 0$  then each point of  $\mathbb{R}^2$  is included in the subset with probability 1. Hence in this degenerate case,  $\lambda_c = 0$  and there cannot be percolation at  $\lambda = \lambda_c$ .  $\square$

We now consider the Random Connection Model, which has parameters  $\lambda$  and  $p$ . In this, points are distributed on the plane according to a homogeneous Poisson point process with rate  $\lambda$ , and then independently for each pair of points  $z_1, z_2$  with  $d(z_1, z_2) < 1$  we declare them to be adjacent with probability  $p$ . Our collection of random subsets of the plane then consists of the connected components of the resulting graph (that is to say, for a given connected component we make a subset of the plane by including every point of the point process, and a straight line connecting each pair of points that are adjacent in the connected component).

In the random connection model, we no longer have that geometrically crossing paths are connected, but for sufficiently high  $p$  the probability that two geometrically crossing paths are connected can be made high.

**Lemma 3.1.6.** *For the RCM there is some  $\varepsilon_1 > 0$  such that for  $p_c > 1 - \varepsilon_1$ , there is no percolation at critical pairs  $(\lambda_c, p_c)$ .*

*Proof.* First we note that the RCM satisfies all conditions to be good, with any  $k > 1$  sufficing. We then need to put a bound on  $p_1$ .

Note that if we have geometrically crossing edges  $(x, y)$  and  $(x', y')$  then by simple geometry, at least two of  $d(x, x'), d(x, y'), d(y, x')$  and  $d(y, y')$  are less than 1. By the independence of edges being included, given that such edges exist the probability that both edges are contained in the same cluster is at least  $1 - (1 - p)^2$ . Since the paths we condition on the existence of in (3.1.1) must cross geometrically at at least one point, we have that given these paths existing

the probability they are in the same component of  $B_{4n+1}$  (and hence probability of  $p_1$ ) is at least  $1 - (1 - p)^2$ .

Then by letting  $p$  be sufficiently close to 1 we see that  $p_1 > p_k$  and our result follows.  $\square$

## 3.2 An Alternative Condition To Rotation Invariance

It is possible to alter this method in order to remove the rotation invariance requirement. Doing so introduces a few additional conditions on the behaviour of crossings of large boxes and requires that any crossing paths connect with high probability, rather than the simple event we considered before; as such we present our results only for models in which paths that cross geometrically are connected. We now present altered conditions for a model without rotation invariance to have a comparable result to that of Theorem 3.1.1.

For  $\rho \geq 1$  we define  $R_{n,\rho} := [-\rho n, \rho n] \times [-n, n]$ ; for a rectangle  $R = [a_1, a_2] \times [b_1, b_2]$  we define  $H(R) := \{a_1, a_2\} \times [b_1, b_2]$  the horizontal edges, and define  $V(R) := [a_1, a_2] \times \{b_1, b_2\}$  the vertical edges.

**Definition 3.2.1.** *We call a planar percolation model 'k-quasigood' if it satisfies the following:*

1. *The model is invariant under reflections of  $P$  in the  $x$  and  $y$  axes.*
2. *The model is invariant under any translation of  $P$ .*
3. *In the model, for  $A \subset A'$ , events of the form  $R \xleftrightarrow[A']{A} R'$  are positively correlated.*
4. *In the model, if two components of the randomly generated subsets have nonempty intersection, they are almost surely the same component and in the same subset.*
5. *When the model percolates the model has an increasing sequence  $(u_n)_{n \in \mathbb{N}}$  such that for all sequences  $(\rho_n)_{n \in \mathbb{N}}$ :*

$$(5a) \lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn, \rho_n}]{!R_{n, \rho_n}!} \partial R_{n, \rho_n} \right) = 1$$



$$(5b) \lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{k_n}]{!B_n!} V(B_n) \right) = 1$$

$$(5c) \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{k_n}]{B_n} H(B_n) \right) > 0$$

6. For any finite subset  $A \subset \mathbb{R}^2$ ,

$$\mathbb{P} \left( \bigcup_{R \in P} A \cap R = \emptyset \right) > 0.$$

7. For any  $A \subset B \subset \mathbb{R}^2$ , any  $\rho \geq 0$ , and any  $n > 0$ , we have

$$\mathbb{P} \left( \bigcap_{R_A \in P_A} \exists R_B \in P_B : R_A \cap R_{n,\rho} \subset R_B \cap R_{n,\rho} \right) = 1$$

Note that any  $k$ -quasigood model which is invariant under rotations by  $\pi/2$  around the origin is also  $k$ -good.

Within the proof of Theorem 3.1.1, the rotation invariance assumption is used primarily in Lemma 3.1.2; we will produce an alternate form that does not rely upon it. For any edge  $A$  which satisfies  $\limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{k_n,\rho}]{R_{n,\rho}} A \right) = 1$ , we can find intervals on that edge much as before which have our desired properties. The potential problems are then first that we have a subsequence with sufficient density that we can find a sub-subsequence with the equivalent of  $a_{3n} < 4a_n$ , and second that we can have a sub-subsequence with the above property for each of the horizontal and vertical edges such that eventually these sub-subsequences are close to each other, and third that we can find some  $\rho$  where the probability of connecting to a horizontal edge is high.

We define

$$\begin{aligned} \bar{p}_h(\rho) &:= \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{k_n,\rho}]{R_{n,\rho}} H(R_{n,\rho}) \right) \\ \bar{p}_v(\rho) &:= \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{k_n,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right). \end{aligned}$$

We begin with showing that we can find sequences comparable to  $(a_n)$  and  $(y_n)$  in these models without rotation invariance, provided the probability of reaching the horizontal edges and the probability of reaching the vertical edges are

both sufficiently large. We can likewise find comparable probabilities for the horizontal edges, however we do not use these and they are only included here for completeness.

**Lemma 3.2.1.** *For a percolating  $k$ -quasigood model, letting  $u_n$  be as in condition (5a), there are  $p_{c,v} < 1$  and  $p_{c,h} < 1$  depending only on the model such that for any  $\rho \geq 1$  with  $\bar{p}_v(\rho) > p_{c,v}$  and  $\bar{p}_h(\rho) > p_{c,h}$ , there exist sequences  $(a_n(\rho))_{n \in \mathbb{N}}$ ,  $(b_n(\rho))_{n \in \mathbb{N}}$ ,  $(y_n(\rho))_{n \in \mathbb{N}}$  and  $(z_n(\rho))_{n \in \mathbb{N}}$  where  $0 < a_n < n$  and  $0 < y_n < n$ , and  $0 < b_n < \rho_n n$  and  $0 < z_n < \rho_n n$  such that for all  $1 > \varepsilon > 0$  we have:*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [a_n - \varepsilon, n] \right) &\geq 1 - (1 - \bar{p}_v(\rho))^{\frac{1}{8}}, \\ \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [y_n - a_n/4 - \varepsilon, y_n + a_n/4 + \varepsilon] \right) &\geq 1 - (1 - \bar{p}_v(\rho))^{\frac{1}{16}}, \\ \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} [b_n - \varepsilon, \rho n] \times \{n\} \right) &\geq 1 - (1 - \bar{p}_h(\rho))^{\frac{1}{8}}, \\ \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} [z_n - b_n/4 - \varepsilon, z_n + b_n/4 + \varepsilon] \times \{n\} \right) &\geq 1 - (1 - \bar{p}_h(\rho))^{\frac{1}{16}}; \end{aligned}$$

where dependence on  $\rho$  has been suppressed.

*Proof.* We can apply a similar proof to that of Lemma 3.1.2. We consider only the vertical edges; the horizontal edges are similar.

We split the set  $V(R_{n,\rho})$  into 4 half edges  $A_i$  as before, such that for all  $i, j$  we have

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} A_i \right) = \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} A_j \right);$$

it then follows from the square root trick immediately that

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} A_i \right) \geq 1 - \left( 1 - \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right) \right)^{\frac{1}{4}}.$$

Should this probability be positive (which by condition (5b) it must for all but finitely many  $n$ ) we can then choose any  $a_n$  and divide  $A_1$  at  $\beta$ , applying the

square root trick to see that

$$\begin{aligned} \max \left( \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [0, \beta] \right), \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [\beta, n] \right) \right) &\geq \quad (3.2.1) \\ &\geq 1 - \left( 1 - \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right) \right)^{\frac{1}{8}}. \end{aligned}$$

Now by condition (6) we see that  $\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [0, 4] \right)$  is bounded by the probability that  $\{0\} \times [0, 4]$  has nonempty intersection with our collection of random subsets by condition (2). As such, we have some  $p_{c,v}$  such that whenever  $1 - \left( 1 - \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right) \right)^{\frac{1}{8}} > p_{c,v}$ , we can find some  $a_n \in [4, n]$  such that whenever  $\beta < a_n$  we have

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [0, \beta] \right) < \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [\beta, n] \right),$$

and whenever  $\beta > a_n$  we have

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [0, \beta] \right) \geq \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [\beta, n] \right).$$

The same argument also shows that in this case  $a_n < n - 4$ . By (3.2.1) we then see that

$$\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [a_n - \varepsilon, n] \right) \geq 1 - \left( 1 - \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right) \right)^{\frac{1}{8}},$$

and by taking the liminf of both sides arrive at

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [a_n - \varepsilon, n] \right) \geq 1 - (1 - \bar{p}_v(\rho))^{\frac{1}{8}}$$

as desired.

We now split  $\{\rho n\} \times [0, a_n + \varepsilon]$  into 2 segments  $S_1 = \{\rho n\} \times [0, a_n/2 + \varepsilon]$  and  $S_2 = \{\rho n\} \times [a_n/2 - \varepsilon, n]$ . If we define  $Y(y, n, \varepsilon) = \{\rho n\} \times [y - a_n/4 - \varepsilon, y + a_n/4 + \varepsilon]$

we can see that

$$\begin{aligned} S_1 &\subset Y(a_n/4) \\ S_2 &\subset Y(3a_n/4 + 1), \end{aligned}$$

and so we consider these as possible values for  $y_n$ . By our construction of  $a_n$ , we see that

$$\begin{aligned} &\max \left( \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} S_1 \right), \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} S_2 \right) \right) \geq \\ &\geq 1 - \left( \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [0, a_n + 2\varepsilon] \right) \right)^{\frac{1}{3}}, \end{aligned}$$

and so by choosing  $y_n$  to maximize  $\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} S_i \right)$  we can apply the same logic as before and take the liminf to obtain

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} \{\rho n\} \times [y_n - a_n/4 - \varepsilon, y_n + a_n/4 + \varepsilon] \right) \geq 1 - (1 - \bar{p}_v(\rho))^{\frac{1}{16}}.$$

□

We shall demonstrate for our  $k$ -quasigood models that we can find arbitrarily large  $\rho$  and infinitely many  $n$  such that we have the event

$$B_{u_n} \xleftrightarrow[R_{4kn,\rho/8+3/4+(\rho/2n,0)}]{R_{4kn,\rho/8+3/4+(\rho/2n,0)}} B_{u_n} + (\rho n, 0)$$

with high probability. We achieve this by connecting together  $\rho$  copies of the box  $B_n$ , and since every step used to connect together 2 squares has high probability, provided we stop after finitely many steps we retain our limiting probability.

**Lemma 3.2.2.** *For a percolating  $k$ -quasigood model, for any increasing sequence  $(n_m)_{m \in \mathbb{N}}$  such that  $a_{3n_m}(1) < 4a_{n_m}(1)$ , and for any  $\alpha \in \mathbb{N}$  we have that*

$$\lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m,\alpha/2+3/4+(2\alpha n_m,0)}]{R_{4kn_m,\alpha/2+3/4+(2\alpha n_m,0)}} B_{u_{3n_m}} + (4\alpha n_m, 0) \right) = 1.$$

*Proof.* We shall prove this by induction on  $\alpha$ . Fix an increasing sequence  $(n_m)$

as desired. Now we assume that for all  $\alpha < M$  we have

$$\lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+3/4+(2\alpha n_m, 0)}]{R_{4n_m, \alpha/2+3/4+(2\alpha n_m, 0)}} B_{u_{3n_m}} + (4\alpha n, 0) \right) = 1.$$

We can now observe that

$$\begin{aligned} & \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4(\alpha+1)n_m, 0) \right) \\ & \geq \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4\alpha n_m, 0) \cap \right. \\ & \quad \cap B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4(\alpha+1)n_m, 0) \cap \\ & \quad \left. \cap B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{!B_{3n_m} + (4\alpha n_m, 0)!} B_{3n_m} + (4\alpha n_m, 0) \right) \\ & \geq \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4\alpha n_m, 0) \right) \times \\ & \quad \times \mathbb{P} \left( B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4(\alpha+1)n_m, 0) \right) + \\ & \quad + \mathbb{P} \left( B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{!B_{3n_m} + (4\alpha n_m, 0)!} B_{3n_m} + (4\alpha n_m, 0) \right) - 1 \end{aligned}$$

since  $3n_m + 4\alpha n_m < 4n_m(\alpha/2 + 5/4) + 2(\alpha+1)n_m = 5n_m + 4\alpha n_m$ .

By condition (7) and event inclusion we see that

$$\begin{aligned} & \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4\alpha n_m, 0) \right) \geq \\ & \geq \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \alpha/2+3/4+(2\alpha n_m, 0)}]{R_{4n_m, \alpha/2+3/4+(2\alpha n_m, 0)}} B_{u_{3n_m}} + (4\alpha n_m, 0) \right), \end{aligned}$$

and by condition (5a) we see that

$$\lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{!B_{3n_m} + (4\alpha n_m, 0)!} B_{3n_m} + (4\alpha n_m, 0) \right) = 1.$$

Then our result follows, provided we can prove that

$$\lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} + (4\alpha n_m, 0) \xleftrightarrow[R_{4kn_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}]{R_{4n_m, \alpha/2+5/4+(2(\alpha+1)n_m, 0)}} B_{u_{3n_m}} + (4(\alpha+1)n_m, 0) \right) = 1.$$

To see this we can apply Corollary 3.1.2 noting that we have all of

$$\begin{aligned} B_{3n_m} + (4\alpha n_m, 0) &\subset R_{4n_m, \alpha/2+5/4} + (2(\alpha+1)n_m, 0), \\ B_{3n_m} + (4(\alpha+1)n_m, 0) &\subset R_{4n_m, \alpha/2+5/4} + (2(\alpha+1)n_m, 0), \\ B_{n_m} + (4\alpha n_m + 2n_m, 3n_m) &\subset R_{4n_m, \alpha/2+5/4} + (2(\alpha+1)n_m, 0). \end{aligned}$$

The proof of Corollary 3.1.2 uses only properties of a model being  $k$ -good that are also satisfied by  $k$ -quasigood models; by our more restrictive condition (4), recalling the definition (3.1.1) we see that  $p_1 = 1$ . Then our result follows.  $\square$

We now produce a collection of lemmas that are intended to show that even if  $\bar{p}_v(\rho) < 1$  we can have the various probabilities related to vertical relations as close to 1 as desired, by letting  $\rho$  be sufficiently high. However we cannot guarantee finding any  $n$  such that both  $a_{3n} < 4a_n$  and  $b_{3n} < 4b_n$ ; nor can we guarantee finding infinitely many  $n < m$  with  $m$  bounded by a constant multiple of  $n$  and both  $a_{3n} < 4a_n$  and  $b_{3n} < 4b_n$ ; as such we cannot use the same method for producing connections between boxes  $B_{u_n}$  for horizontal and for vertical translations as we did before.

In order to prove our eventual result, we would ideally be able to find some  $\rho > 0$  such that  $\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn, \rho}]{R_{n, \rho}} V(R_{n, \rho}) \right) = 1$ ; however it suffices to show that for any  $p < 1$  we can find  $\rho \geq 1$  such that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn, \rho}]{R_{n, \rho}} V(R_{n, \rho}) \right) > p.$$

In order to prove this, we take a large number of identical copies  $B_n$  each at horizontal distance at least  $2kn$  from each other, such that the events that  $B_{u_n} \xleftrightarrow[B_{kn}]{B_n} \partial B_n$  are all independent; if we take enough copies then the probability at least one of them contains a single component that touches both the top and bottom edge of  $B_n$  becomes close to 1. We can then also use Lemma 3.2.2 so that for any fixed integer  $\rho'$  we can find infinitely many  $m$  such that we have a path from  $B_{u_m}$  to the right hand edge of  $R_{m, \rho'}$  with probability close to 1. Then we see that the probability that there is a path from  $B_{u_n}$  to the top edge is at least the probability we can find a path from  $B_{u_n}$  that crosses at least  $\ell$  boxes  $B_n$  each at distance at least  $2kn$  from each other and that one of those boxes

contains a vertical crossing.

**Lemma 3.2.3.** *For a percolating  $k$ -quasigood model, for any  $\beta \in \mathbb{N}$ , for any  $1 > \varepsilon > 0$ , we have*

$$\lim_{\rho \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn,\rho}]{R_{n,\rho}} V(R_{n,\rho}) \right) = 1, \quad (3.2.2)$$

and moreover that for any sequence  $n_m$  where  $a_{3n_m} < 4a_{n_m}$  and any sequence  $n'_m$  with  $n_m \leq n'_m \leq 15n_m$  we have that

$$\lim_{\rho \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{n'_m}} \xleftrightarrow[R_{kn'_m,\rho}]{R_{n'_m,\rho}} V(R_{n'_m,\rho}) \right) = 1.$$

*Proof.* Let  $n'$  be fixed and consider the event

$$HH(B_{n'}) := \left( B_{u_{n'}} \xleftrightarrow[B_{kn'}]{B_{n'}} [-n, n] \times \{n\} \right) \cap \left( B_{u_{n'}} \xleftrightarrow[B_{kn'}]{B_{n'}} [-n, n] \times \{-n\} \right).$$

We can easily see both that

$$\mathbb{P} \left( B_{u_{n'}} \xleftrightarrow[B_{kn'}]{B_{n'}} H(B_{n'}) \right) \geq \mathbb{P}(HH(B_{n'}))$$

and that

$$\begin{aligned} \mathbb{P}(HH(B_{n'})) &\geq \mathbb{P} \left( B_{u_{n'}} \xleftrightarrow[B_{kn'}]{B_{n'}} [-n, n] \times \{n\} \right)^2 \\ &\geq \left( 1 - \left( 1 - \mathbb{P} \left( B_{u_{n'}} \xleftrightarrow[B_{kn'}]{B_{n'}} H(B_{n'}) \right) \right)^{1/2} \right)^2. \end{aligned}$$

Now we let  $c = \liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[B_{kn}]{B_n} H(B_n) \right) > 0$ , set  $p < 1$ , and choose  $\alpha$  a multiple of 27 such that  $1 - (1 - c^2)^{\alpha/27} > p$ .

We now choose an increasing sequence  $n_m$  such that  $a_{3n_m}(1) < 4a_{n_m}(1)$  and any sequence  $n'_m$  with  $n_m \leq n'_m \leq 15n_m$ ; we can find such a sequence  $n_m$  by Corollary 3.1.1. Now we consider the situation that have all of the following simultaneously

- $B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{4n_m, 1+4\alpha k}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0);$

- for each  $\beta \in \mathbb{N}$ ,  $1 \leq \beta \leq \alpha/27 - 1$  we have that

$$B_{u_{3n'_m}} + (7\beta kn'_m, 0) \xleftrightarrow[B_{3kn'_m} + (7\beta kn'_m, 0)]{!B_{3n'_m} + (7\beta kn'_m, 0)!} \partial B_{3n'_m} + (7\beta kn'_m, 0);$$

- there is at least one  $0 \leq \beta \leq \alpha/27 - 1$  such that we have

$$HH(B_{3n'_m} + (7\beta kn'_m, 0)).$$

We now break into cases. In the first case, the path

$$B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{4n_m, 1+4\alpha k}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0)$$

contains points outside  $R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}}$ . Since  $n_m \leq n'_m$  we have  $u_{3n_m} \leq u_{3n'_m}$ , and then by event inclusion we have

$$B_{u_{3n'_m}} \xleftrightarrow[R_{3kn'_m, 4(1+4\alpha k)/3}]{R_{3n'_m, 4(1+4\alpha k)/3}} V(R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}).$$

In the second case, there is a path

$$B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0);$$

and we see that this path must contain a horizontal crossing of each box  $B_{3n'_m} + (6\beta kn'_m, 0)$  where  $(7\beta k + 3)n'_m \leq 4(\alpha k + 1)n_m - u_{3n_m}$  and  $(7\beta k - 3)n'_m \geq u_{3n_m}$ . Since  $u_{3n} \leq n$  we see that the first inequality is satisfied provided

$$(7\beta k + 3)n'_m \leq (4\alpha k)n_m$$

$$\beta \leq 4\alpha \frac{n_m}{7n'_m} - \frac{3}{7k}.$$

Then since  $k \geq 1$  and  $n_m/n'_m \geq 1/15$  we see that this will hold whenever

$$\beta \leq \alpha/27 - 1 \leq 4\alpha/105 - 1.$$



The second inequality holds provided

$$(7\beta k - 3)n'_m \geq n_m \geq u_{3n_m}$$

$$\beta \geq \frac{n_m}{7kn'_m} + \frac{3}{7k},$$

and thus since  $n_m/n'_m \leq 1$  this inequality holds whenever  $\beta \geq 1 \geq 1/7k + 3/7k$ .

If  $HH(B_{3n'_m})$  occurs, then we immediately have a path

$$B_{u_{3n'_m}} \xleftrightarrow[R_{3kn'_m, 4(1+4\alpha k)/3}]{R_{3n'_m, 4(1+4\alpha k)/3}} V(R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}).$$

Finally we note that if for some  $1 \leq \beta \leq \alpha/27 - 1$  all of

$$B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0),$$

$$B_{u_{3n'_m}} + (7\beta kn'_m, 0) \xleftrightarrow[B_{3kn'_m} + (7\beta kn'_m, 0)]{!B_{3n'_m} + (7\beta kn'_m, 0)!} \partial B_{3n'_m} + (7\beta kn'_m, 0),$$

$$HH(B_{3n'_m} + (7\beta kn'_m, 0))$$

occur then by condition (7) there must be a horizontal and vertical crossing of the box  $B_{3n'_m} + (7\beta kn'_m, 0)$  in the model restricted to  $R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}}$ . As such, by condition (4) the paths producing crossings are connected. Since the horizontal crossing is produced by the path

$$B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0),$$

$B_{u_{3n_m}} \subset B_{u_{3n'_m}}$ , and by event inclusion we must then have a path

$$B_{u_{3n'_m}} \xleftrightarrow[R_{3kn'_m, 4(1+4\alpha k)/3}]{R_{3n'_m, 4(1+4\alpha k)/3}} V(R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}).$$

Since  $\frac{n_m}{n'_m} \leq 1$  we have

$$R_{3n'_m, 4(1+4\alpha k)/3 \times \frac{n_m}{n'_m}} \subset R_{3n'_m, 4(1+4\alpha k)/3},$$

and so we have that

$$\begin{aligned}
& \mathbb{P} \left( B_{u_{3n'_m}} \xleftrightarrow[R_{3kn'_m, 4(1+4\alpha k)/3}]{R_{3n'_m, 4(1+4\alpha k)/3}} V(R_{3n'_m, 4(1+4\alpha k)/3}) \right) \geq \\
& \geq \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{4n_m, 1+4\alpha k}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0) \cap \right. \\
& \quad \bigcap \bigcap_{\beta=1}^{\alpha/27-1} B_{u_{3n'_m}} + (7\beta kn'_m, 0) \xleftrightarrow[B_{kn'_m} + (7\beta kn'_m, 0)]{!B_{3n'_m} + (7\beta kn'_m, 0)!} \partial B_{3n'_m} + (7\beta kn'_m, 0) \cap \\
& \quad \left. \bigcap \bigcup_{\beta=1}^{\alpha/27-1} HH(B_{3n'_m} + (7\beta kn'_m, 0)) \right) \geq \\
& \geq \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, 1+4\alpha k}]{R_{4n_m, 1+4\alpha k}} B_{u_{3n_m}} + (4(\alpha k + 1)n_m, 0) \right) \times \\
& \quad \times \mathbb{P} \left( \bigcup_{\beta=0}^{\alpha/27-1} HH(B_{3n'_m} + (7\beta kn'_m, 0)) \right) + \\
& \quad + (\alpha/27 - 1) \left( \mathbb{P} \left( B_{u_{3n'_m}} \xleftrightarrow[B_{kn'_m}]{!B_{3n'_m}!} \partial B_{3n'_m} \right) - 1 \right).
\end{aligned}$$

where we arrive at the final inequality using the union bound and conditions (3) and (2).

Note that the events  $HH(B_{3n'_m} + (7\beta kn'_m, 0))$  and  $HH(B_{3n'_m} + (7\beta' kn'_m, 0))$  are independent whenever  $|\beta - \beta'| \geq 1$ . Since there are  $\alpha/27 - 1$  different integer values satisfying our inequalities, by condition (2) we see that

$$\mathbb{P} \left( \bigcup_{\beta=0}^{\alpha/27-1} HH(B_{3n'_m} + (7\beta kn'_m, 0)) \right) = 1 - (1 - \mathbb{P}(HH(B_{3n'_m})))^{\alpha/27} > p$$

for sufficiently high  $m$ . The result then follows from Lemma 3.2.2, condition (5a), and  $\mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn, \rho}]{R_{n, \rho}} V(R_{n, \rho}) \right)$  being increasing in  $\rho$ .  $\square$

While we can connect rectangles horizontally and vertically using the same method as in the rotation invariant case for both, this does not give us fine control over the relative length of the connections; in order to achieve our result however we need these lengths to be within a constant multiple of each other. However, thanks to our stronger condition (4), we have an alternative method to demonstrate that connections between vertically displaced rectangles occur, by constructing the same sort of event in the overlap as is used in the method of Bollobás and Riordan. Rather than finding a square as before, we find a rectangle long enough that the section of the rectangle between the boxes  $B_{u_n}$  is longer

than our chosen  $\rho$ , and use Lemma 3.2.2 to show that there is a long way crossing of the region  $R$  between  $R_{m,\rho}$  and  $R_{m,\rho} + (0, \ell)$ . Then if we have a path from  $B_{u_m}$  to the top of  $R_{m,\rho}$  and a path from  $B_{u_m} + (0, \ell)$  to the bottom of  $R_{m,\rho} + (0, \ell)$ , these both contain vertical crossings of  $R$  and hence are both connected to the known horizontal crossing of  $R$ , and hence to each other. In this way we can stitch boxes together vertically without implementing any controls on  $b_n$ .

**Lemma 3.2.4.** *For an increasing sequence  $n_m$  such that  $a_{3n_m} < 4a_{n_m}$ , we have that*

$$\liminf_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{15n_m}} \xleftrightarrow[R_{36kn_m, \rho} + (0, 11n_m)]{R_{36n_m, \rho} + (0, 11n_m)} B_{u_{15n_m}} + (0, 22n_m) \right) \geq (1 - (1 - \bar{p}_v(\rho))^{1/2})^2.$$

*Proof.* We fix our sequence  $n_m$  and fix  $\rho \geq 1$ . By Lemma 3.2.2, for any  $\alpha \in \mathbb{N}$  we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} + (-4\alpha, 0) \xleftrightarrow[R_{4kn_m, 1+\alpha}]{R_{4n_m, 1+\alpha}} B_{u_{3n_m}} + (4\alpha, 0) \right) = 1.$$

Now we let  $\alpha = \lceil 15\rho/4 + 1 \rceil$  and consider the event that the following occur simultaneously

- $B_{u_{15n_m}} \xleftrightarrow[R_{15kn_m, \rho}]{R_{15n_m, \rho}} [-15\rho n_m, 15\rho n_m] \times \{15n_m\},$
- $B_{u_{15n_m}} + (0, 22n_m) \xleftrightarrow[R_{15kn_m, \rho} + (0, 22n_m)]{R_{15n_m, \rho} + (0, 22n_m)} [-15\rho n_m, 15\rho n_m] \times \{7n_m\},$
- $B_{u_{3n_m}} + (-4\alpha n_m, 11n_m) \xleftrightarrow[R_{4kn_m, 1+\alpha} + (0, 11n_m)]{R_{4n_m, 1+\alpha} + (0, 11n_m)} B_{u_{3n_m}} + (4\alpha n_m, 11n_m).$

We note that since  $k \geq 1$  we have that

$$R_{15kn_m, \rho} + (0, 22kn_m) \subset R_{36kn_m, \rho} + (0, 11kn_m);$$

additionally, since  $4kn_m(1 + \alpha) \leq (15\rho + 8)kn_m \leq 36\rho kn_m$  we have

$$R_{4kn_m, 1+\alpha} + (0, 11n_m) \subset R_{36kn_m, \rho} + (0, 11kn_m).$$

Since the right hand edge of  $B_{u_{3n_m}} + (-4\alpha n_m, 11n_m)$  has  $x$  coordinate at most  $n_m - 4\alpha n_m \leq n_m - (15\rho + 4)n_m < 15\rho n_m$ , and by symmetry the left hand edge of

$B_{u_{3n_m}} + (4\alpha, 11n_m)$  has  $x$  coordinate at least  $4\alpha n_m - n_m \geq (15\rho + 3)n_m > 15\rho n_m$  we see that should the event

$$B_{u_{3n_m}} + (-4\alpha n_m, 11n_m) \xleftrightarrow[R_{4kn_m, 1+\alpha+(0, 11n_m)}]{R_{4n_m, 1+\alpha+(0, 11n_m)}} B_{u_{3n_m}} + (4\alpha n_m, 11n_m)$$

occur there must be a horizontal crossing of the box  $R_{4n_m, 15\rho/4} + (0, 11n_m) = [-15\rho n_m, 15\rho n_m] \times [7n_m, 15n_m]$ .

Similarly, since  $u_{15n_m} \leq 5n_m$  we see that should the event

$$B_{u_{15n_m}} \xleftrightarrow[R_{15kn_m, \rho}]{R_{15n_m, \rho}} [-15\rho n_m, 15\rho n_m] \times \{15n_m\}$$

occur then in the model restricted to  $R_{15kn_m, \rho}$  we must have a component which contains a point in  $B_{u_{15n_m}}$  and contains a vertical crossing of the box

$$[-15\rho n_m, 15\rho n_m] \times [7n_m, 15n_m];$$

by reflection we also see that should the event

$$B_{u_{15n_m}} + (0, 22n_m) \xleftrightarrow[R_{15kn_m, \rho} + (0, 22n_m)]{R_{15n_m, \rho} + (0, 22n_m)} [-15\rho n_m, 15\rho n_m] \times \{7n_m\}$$

occur then in the model restricted to  $R_{15kn_m, \rho} + (0, 22n_m)$  we must have a component which contains a point in  $B_{u_{15n_m}} + (0, 22n_m)$  and contains a vertical crossing of the box  $[-15\rho n_m, 15\rho n_m] \times [7n_m, 15n_m]$ .

Then since a horizontal and a vertical crossing of the same box must cross geometrically we see that should all these events occur simultaneously then by conditions (4) and (7), and by the set inclusions noted above we have the event

$$B_{u_{15n_m}} \xleftrightarrow[R_{36kn_m, \rho} + (0, 11n_m)]{R_{36n_m, \rho} + (0, 11n_m)} B_{u_{15n_m}} + (0, 22n_m).$$

We note that by condition (3) the three events are positively correlated. Thus

we see that

$$\begin{aligned}
& \mathbb{P} \left( B_{u_{15n_m}} \xleftrightarrow{R_{36n_m, 15\rho/36+1/3+(0, 11n_m)}} B_{u_{15n_m}} + (0, 22n_m) \right) \geq \\
& \geq \mathbb{P} \left( B_{u_{15n_m}} \xleftrightarrow{R_{15n_m, \rho}} [-15\rho n_m, 15\rho n_m] \times \{15n_m\} \right)^2 \times \\
& \times \mathbb{P} \left( B_{u_{3n_m}} + (-4\alpha n_m, 11n_m) \xleftrightarrow{R_{4n_m, 1+\alpha+(0, 11n_m)}} B_{u_{3n_m}} + (4\alpha n_m, 11n_m) \right).
\end{aligned}$$

Now by Lemma 3.2.2 we see that

$$\lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} + (-4\alpha n_m, 11n_m) \xleftrightarrow{R_{4n_m, 1+\alpha+(0, 11n_m)}} B_{u_{3n_m}} + (4\alpha n_m, 11n_m) \right) = 1,$$

and by definition of  $\bar{p}_v(\rho)$  and applying the square root trick we see that for  $m$  sufficiently large

$$\mathbb{P} \left( B_{u_{15n_m}} \xleftrightarrow{R_{15n_m, \rho}} [-15\rho n_m, 15\rho n_m] \times \{15n_m\} \right) \geq 1 - (1 - \bar{p}_v(\rho))^{1/2},$$

and our result follows by taking the limit infimum as  $m \rightarrow \infty$ .  $\square$

The idea used here is the same as that we used to prove Theorem 3.1.1, with the slight added complication that instead of having one rule to judge if an edge should be included, we have separate rules for horizontal and vertical edges. Using the previous lemmas we can ensure that the events that a horizontal edge is included and that a vertical edge is included both depend on finite boxes, with the relative size of the boxes bounded by a constant. Using this constant bound on the relative size, we can produce a collection of linked finite range dependent models which all have the same finite range; from there the methods are much the same.

**Theorem 3.2.1.** *For a  $k$ -quasigood planar percolation model, there is some  $p_k < 1$  and a pair of collections of events  $E(R_{n,\rho})$  and  $E'(R_{n',\rho'})$  which each depend only on the state of the point process within  $R_{n,\rho}$  and  $R_{n',\rho'}$  respectively*

such that should we have some  $n$  and,  $\rho \geq \rho' \geq 1$  with

$$\begin{aligned}\mathbb{P}(E(R_{n,\rho})) &> p_k, \\ \mathbb{P}(E'(R_{6n,\rho'})) &> p_k,\end{aligned}$$

then the model percolates; and whenever the model percolates we can find  $n, \rho$  and  $\rho'$  satisfying these equations. Consequently, should we additionally have that the state of the point process within a fixed box  $R_{n,\rho}$  be continuous in the parameters of the process, we have that the set of parameters for which there is percolation is open.

*Proof.* We proceed in a similar fashion to the proof of Theorem 3.1.1. In order to achieve this, we produce a coupled discrete bond percolation model with finite range dependence.

For any  $n, \rho_1, \rho_2$  with  $\rho_1 \geq \rho_2 \geq 1$  and  $6\rho_1 > 22$ , we can produce a bond percolation model on  $\mathbb{Z}^2$ . We let  $z(x, y) = ((6\rho_1 - 2)nx, 22ny)$  and include the edge  $\{(x, y), (x + 1, y)\}$  whenever the following all hold:

$$\begin{aligned}B_{u_{3n}} + z(x, y) &\xleftrightarrow[R_{6kn, \rho_1} + z(x+1/2, y)]{R_{6n, \rho_1} + z(x+1/2, y)} B_{u_{3n}} + z(x + 1, y), \\ B_{u_{3n}} + z(x, y) &\xleftrightarrow[B_{3kn} + z(x, y)]{!B_{3n} + z(x, y)!} \partial B_{3n} + z(x, y) \\ B_{u_{3n}} + z(x + 1, y) &\xleftrightarrow[B_{3kn} + z(x+1, y)]{!B_{3n} + z(x+1, y)!} z(x + 1, y) + \partial B_{3n} + z(x + 1, y),\end{aligned}$$

and including the edge  $\{(x, y), (x, y + 1)\}$  whenever the following all hold:

$$\begin{aligned}B_{u_{15n}} + z(x, y) &\xleftrightarrow[R_{36kn, \rho_2} + z(x, y+1/2)]{R_{36n, \rho_2} + z(x, y+1/2)} B_{u_{15n}} + z(x, y + 1), \\ B_{u_{15n}} + z(x, y) &\xleftrightarrow[B_{15kn} + z(x, y)]{!B_{15n} + z(x, y)!} \partial B_{15n} + z(x, y) \\ B_{u_{15n}} + z(x, y + 1) &\xleftrightarrow[B_{15kn} + z(x, y+1)]{!B_{15n} + z(x, y+1)!} \partial B_{15n} + z(x, y + 1).\end{aligned}$$

We let  $E(R_{6kn, \rho_1} + z(1/2, 0))$  be the event that the edge  $\{(0, 0), (1, 0)\}$  is included, and let  $E'(R_{36kn, \rho_2} + z(0, 1/2))$  be the event that the edge  $\{(0, 0), (0, 1)\}$  is included.

Each of these edges depends only on the arrivals within some finite rectan-

gle, and so two edges are independent if their corresponding rectangles do not intersect. All these finite rectangles are contained in appropriate translations of  $R_{\text{lim}} = R_{36kn, \rho_1}$ . Then since moving vertically one edge corresponds to translating the associated rectangle by  $(0, 11n)$  we see that sites are independent if at vertical distance at least  $7k$  (since  $R_{\text{lim}}$  spans vertical distance  $72kn$ ); similarly moving horizontally by one edge translates the associated rectangle by  $((6\rho_1 - 2)n, 0)$  and so edges are independent if at horizontal distance at least  $\frac{72kn\rho_1}{(6\rho_1 - 2)n} \leq 18k$ . Then we see that the discrete model has edges independent if at graph distance at least  $25k$ , and thus is  $25k$ -dependent. We then let  $p_k < 1$  be such that any  $25k$ -dependent model percolates provided the marginal probability of an edge being included is at least  $p_k$ ; hence if  $\mathbb{P}(E(R_{6kn, \rho_1} + z(1/2, 0))) > p_k$  and  $\mathbb{P}(E'(R_{36kn, \rho_2} + z(0, 1/2))) > p_k$ .

Now we assume that the model percolates. Given  $p < 1$ , by Lemma 3.2.3 we can choose  $\rho'(p) \geq 1$  such that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left( B_{u_n} \xleftrightarrow[R_{kn, \rho'(p)}]{R_{n, \rho'(p)}} V(R_{n, \rho'(p)}) \right) > p. \quad (3.2.3)$$

We can then choose a sequence  $(n_m)$  such that  $u_{3n_m} < 4u_{n_m}$  and by Lemma 3.2.4 we see that

$$\liminf_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{15n_m}} \xleftrightarrow[R_{36kn_m, \rho'(p) + (0, 11n_m)}]{R_{36n_m, \rho'(p) + 1/3 + (0, 11n_m)}} B_{u_{15n_m}} + (0, 22n_m) \right) > (1 - (1 - p)^{1/2})^2.$$

Then we choose  $p > 1 - (1 - p_k^{1/2})^2$ , so that  $(1 - (1 - p)^{1/2})^2 > p_k$ , set  $\rho_2 = \rho'(p)$ , and set  $\rho_1 = \lceil 6\rho_2 \rceil$ , noting that since  $\rho_2 \geq 1$  we have that  $6\rho_1 \geq 36$ . Since  $R_{4kn_m, 1+\alpha} \subset R_{6kn_m, \alpha}$  and  $R_{4n_m, 1+\alpha} \subset R_{6n_m, \alpha}$ , by Lemma 3.2.2 we have both

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{4kn_m, \rho_1 + 1 + (2\rho_1 n_m)}]{R_{4n_m, \rho_1 + 1 + (2\rho_1 n_m)}} B_{u_{3n_m}} + (4\rho_1 n_m, 0) \right) &= 1, \text{ and} \\ \lim_{m \rightarrow \infty} \mathbb{P} \left( B_{u_{3n_m}} \xleftrightarrow[R_{6kn_m, \rho_1 + (2\rho_1 n_m)}]{R_{6n_m, \rho_1 + (2\rho_1 n_m)}} B_{u_{3n_m}} + (4\rho_1 n_m, 0) \right) &= 1. \end{aligned}$$

Finally since by condition (5a)

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( B_{u_{3n}} + z(x, y) \xleftrightarrow[B_{3kn + z(x, y)}]{!B_{3n + z(x, y)!}} \partial B_{3n} + z(x, y) \right) = 1$$

for all  $n$  we see that whenever the model percolates we can find some  $\rho_1 \geq \rho_2$  with  $6\rho_1 > 22$  and some  $n$  sufficiently large that  $\mathbb{P}(E(R_{6kn, \rho_1} + z(1/2, 0))) > p_k$  and  $\mathbb{P}(E'(R_{36kn, \rho_2} + z(0, 1/2))) > p_k$ .

Lastly we need to prove that should some coupled discrete model percolate then the original model must also percolate. We fix  $n$ ,  $\rho_1$  and  $\rho_2$ , and claim that should  $(x_1, y_1)$  and  $(x_2, y_2)$  be connected in the discrete model, then there is a component of the original model with nonempty intersection with both  $B_{u_{15n}} + z(x_1, y_1)$  and  $B_{u_{15n}} + z(x_2, y_2)$ . To see this, we work by induction on the minimum path length in the discrete model between  $(x_1, y_1)$  and  $(x_2, y_2)$  using only edges that were accepted. Should this path length be 1 then we see that either  $x_2 = x_1 + 1$  and  $y_2 = y_1$  or vice versa. We assume the first; the second case is similar. Then since  $\{(x_1, y_1), (x_1 + 1, y_1)\}$  was included in our discrete model, by definition we have that

$$B_{u_{3n}} + z(x_1, y_1) \xleftrightarrow[R_{6kn, \rho_1} + z(x_1 + 1/2, y_1)]{R_{6n, \rho_1} + z(x_1 + 1/2, y_1)} B_{u_{3n}} + z(x_1 + 1, y_1),$$

and then since  $u_{3n} \leq u_{15n}$  and by condition (7) we have our claim.

Now assume our claim holds for all paths of length less than  $M$ , and let  $(x_2, y_2)$  be at path length  $M$  from  $(x_1, y_1)$ . We now consider cases; first the case that there is some  $(x, y) \neq (x_1, y_1)$  and  $(x, y) \neq (x_2, y_2)$  which is on a minimal path between  $(x_1, y_1)$  and  $(x_2, y_2)$  such that  $(x, y)$  is an endpoint for a vertical edge included in our discrete model. Then by definition we have that

$$B_{u_{15n}} + z(x, y) \xleftrightarrow[B_{15kn} + z(x, y)]{!B_{15n} + z(x, y)!} \partial B_{15n} + z(x, y).$$

Now by our assumption we have components  $C_1$  and  $C_2$  of the model such that both  $C_1$  and  $C_2$  contain a point of  $B_{u_{15n}} + z(x, y)$ , that  $C_1$  contains a point of  $B_{u_{15n}} + z(x_1, y_1)$ , and that  $C_2$  contains a point of  $B_{u_{15n}} + z(x_2, y_2)$ . Note that  $u_{15n} \leq 5n$  and that  $z(1, 0) = ((6\rho_1 - 2)n, 0)$ , so the right hand edge of  $B_{u_{15n}} + z(x - 1, y)$  is at distance at least  $(6\rho_1 - 2)n - 20n > 0$  from the left hand edge of  $B_{15n} + z(x, y)$ ; similarly the top edge of  $B_{u_{15n}} + z(x, y - 1)$  is at distance at least  $2n$  from the bottom edge of  $B_{15n} + z(x, y)$ , so for  $(x', y') \neq (x, y)$  we have that  $B_{15n} + z(x, y) \cap B_{u_{15n}} + z(x', y') = \emptyset$ . Since  $(x, y)$  is connected to a vertical



edge by assumption, we have that

$$B_{u_{15n}} + z(x, y) \xleftrightarrow[B_{15kn} + z(x, y)]{!B_{15n} + z(x, y)!} \partial B_{15n} + z(x, y),$$

then since both  $C_1$  and  $C_2$  contain points outside  $B_{15n} + z(x, y)$  we must have that they are connected in our model, and we have our claim.

If no such  $(x, y)$  exists, then without loss of generality we see that  $(x_2, y_2) = (x_1 + M, y_1)$  and that we have each of the edges  $\{(x_1 + i, y_1), (x_1 + i + 1, y_1)\}$  for  $0 \leq i \leq M - 1$ . In this case we claim additionally that we have a component of the model which has nonempty intersection with  $B_{u_{3n}} + z(x_1 + i, y_1)$  for all  $0 \leq i \leq M - 1$ . We will once more perform induction on  $M$ . As before our claim is immediate for the case  $M = 2$  by the definition of an edge being included; we then assume our claim is true for  $M < M'$ . Then let us have a collection of edges  $\{(x_1 + i, y_1), (x_1 + i + 1, y_1)\}$  for  $0 \leq i \leq M' - 1$ . We have by assumption that we have a component of the model  $C_1$  which has nonempty intersection with  $B_{u_{3n}} + z(x_1 + i, y_1)$  for all  $0 \leq i \leq M' - 2$ . Additionally, since the edge  $\{(x_1 + M' - 1, y_1), (x_1 + M', y_1)\}$  is included in the discrete model it must be the case that

$$B_{u_{3n}} + z(x_1 + M' - 1, y_1) \xleftrightarrow[R_{6kn, \rho_1} + z(x_1 + 1/2, y)]{R_{6n, \rho_1} + z(x_1 + 1/2, y)} B_{u_{3n}} + z(x_1 + M', y_1),$$

so by condition (7) we see that we have a component  $C_2$  in our model which has both a point of  $B_{u_{3n}} + z(x_1 + M' - 1, y_1)$  and a point of  $B_{u_{3n}} + z(x_1 + M', y_1)$ . Then since as with the previous case  $B_{15n} + z(x, y) \cap B_{u_{15n}} + z(x', y') = \emptyset$  for  $(x, y) \neq (x', y')$  and since  $B_{3n} \subset B_{15n}$  and  $B_{u_{3n}} \subset B_{u_{15n}}$  we see that since

$$B_{u_{3n}} + z(x_1 + M' - 1, y_1) \xleftrightarrow[B_{3kn} + z(x_1 + M' - 1, y_1)]{!B_{3n} + z(x_1 + M' - 1, y_1)!} \partial B_{3n} + z(x_1 + M' - 1, y_1)$$

and since both  $C_1$  and  $C_2$  contain points outside  $B_{3n}$ , we must have that  $C_1$  and  $C_2$  are connected and are thus by condition (7) part of the same component of the model as a whole.

Hence if we have an unbounded component in any coupled discrete percolation model, we have percolation in the original model.  $\square$

While conditions (5b) and (5c) are simple to prove for models which are equiv-

alent to rescaled rotation invariant  $k$ -good models, due to the ease of transforming the model into a rotation invariant form, we do not benefit by our theorem in this case. Conditions (5b) and (5c) seem to be believable requirements, however no immediate proof of these conditions for a nontrivial non-rotation invariant model presents itself.

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